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DIPLOMOVÁ PRÁCE
MASTER'S THESIS

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SOLUTIONS OF INVERSE PROBLEMS IN THE AREA OF MATERIAL AND HEAT EXCHANGERS

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AUTHOR

VEDOUCÍ PRÁCE — doc. RNDr. JAN ČERMÁK, CSc.
SUPERVISOR

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ZADÁNÍ DIPLOMOVÉ PRÁCE

student(ka): Bc. Tereza Brožová

který/která studuje v **magisterském navazujícím studijním programu**

obor: **Matematické inženýrství (3901T021)**

Ředitel ústavu Vám v souladu se zákonem č.111/1998 o vysokých školách a se Studijním a zkušebním řádem VUT v Brně určuje následující téma diplomové práce:

Řešení inverzních úloh v oblasti výměníků hmoty a tepla

v anglickém jazyce:

Solutions of inverse problems in the area of material and heat exchangers

Stručná charakteristika problematiky úkolu:

Matematický model dynamického chování výměníků hmoty a tepla lze standardně popsat pomocí soustavy diferenciálních rovnic. Kvalitativní analýza tohoto modelu vyžaduje zodpovězení základních otázek týkajících se existence a jednoznačnosti řešení, příp. jeho stability. Zodpovězení těchto otázek umožňuje řešení inverzní úlohy, tj. určení některý parametrů výměníku hmoty nebo tepla na základě znalosti modelu a některý jeho měřitelných veličin.

Cíle diplomové práce:

Cílem diplomové práce, která navazuje na bakalářskou práci, by mělo být zpřesnění rešerší matematického popisu dynamického chování výměníků hmoty a tepla, založeném na systému diferenciálních rovnic. V této souvislosti bude práce řešit složitější model, ve kterém nejsou zanedbány časové změny teploty, příp. koncentrací. V případě složitějšího modelu zahrnujícího časovou proměnnou se předpokládá provedení analýzy jeho řešitelnosti a jednoznačnosti, příp. stability řešení inverzních úloh. Práce by dále měla diskutovat otázky měřitelnosti parametrů získaných řešením inverzní úlohy, a také otázky kontrolovatelnosti a měřitelnosti parametrů, které zkoumaný model popisují.

Summary

This master's thesis deals with the dynamic behaviour of the heat exchangers which is described by a system of differential equations. In this connection, it contains general informations about heat transfer, heat exchangers and their arrangements. The main aim of this thesis is to solve the inverse problem of the antiparallel arrangement and discuss the question of the controllability, observability and identifiability of its parameters.

Keywords

heat transfer, heat exchange, identifiability, controllability, observability, inverse problem

Abstrakt

Tato diplomová práce se zabývá dynamickým chováním výměníků tepla, které je popsáno systémem diferenciálních rovnic. V této souvislosti obsahuje obecné informace o přenosu tepla, výměnících tepla a jejich uspořádání. Hlavním cílem této práce je řešení inverzní úlohy protiproudého uspořádání a diskuze otázky řiditelnosti, pozorovatelnosti a identifikovatelnosti jeho parametrů.

Klíčová slova

prostup tepla, výměna tepla, identifikovatelnost, řiditelnost, pozorovatelnost, inverzní úloha

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Declaration

I declare that I have written my master's thesis on the theme of "Solutions of inverse problems in the area of material and heat exchangers" independently, under the guidance of the master's thesis supervisor and using the technical literature and other sources of information which are all quoted in the thesis and detailed in the list of literature at the end of the thesis.

Bc. Tereza Brožová

Acknowledgement

I would like to thank to RNDr. Jan Krejčí, PhD. for his professional leadership, valuable comments and for his expert advices during compile this master's thesis.
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My special thanks belong to my parents Zdeňka and Vojtěch Broža for their love, patience, support and for allowing me to study. Thanks mom and dad.

Bc. Tereza Brožová

Contents

1 Introduction

This master's thesis follows the bachelor thesis "Solution of Inverse Problems in the Area of Material and Heat Exchangers" (see [1]). It investigates a similar system, however, unlike the bachelor work, this thesis deals with the system depending on time. The studied problem is described by a system of differential equations. Because of the time dependence, this problem is much more complex and includes, among others, time variations of temperature, respectively concentration (in the case of material exchangers).

As an example of this system, a heat exchanger or a chemical reactor can serve. In the case of a heat exchanger, a change of energy occurs, and in the case of a chemical reactor, the concentration of a substance is changed. We can characterize the studied system by an input flow of a refrigerated liquid Q_{SI} and by its output flow Q_{SO}. Analogously, we consider the coolant Q_{DI}, Q_{DO}. In the case of a heat exchanger, we involve also input temperature of a refrigerated liquid T_{SI} and its output temperature T_{SO}. Analogously, we introduce the symbols T_{DI}, T_{DO} for the coolant. If we consider chemical reactor, then we add the concentrations of substances C_{SI}, C_{SO}, C_{DI}, C_{DO}, where the meaning of the symbols I and O is the same as above, S denotes a sample and D is a dialysate.

The main goal of this thesis is to determine the temperature of refrigerated liquid S, respectively its concentration, from the measured data in the coolant area D. First, we should answer the question of importance of this topic. We can imagine the studied model as a microdialysis probe, or a cooler of a nuclear reactor, or air conditioning, or condensers, or heaters, or vacuum applications. In the case of fluid, the situation is much more complicated because its behaviour can depend on pressure (gasses are not incompressible). For this reason, we deal only with the case of liquid.

The studied problem is solved by the inverse problem method for the following reason. Sometimes it can happen that the circuit of sample is inaccessible like it is in the case of microdialysis (the sample is inside the human body). Sometimes we even do not wish to be in contact with the sample. For example, if we consider the cooler of a nuclear reactor, then the refrigerated liquid is very radioactive and it could be dangerous for a human to be exposed to radioactivity effects. It can also hapen that the device is so small that it is impossible to measure anything in the circuit of the sample. Also, it could be impossible to measure anything inside the compartment of the refrigerated liquid from the construction point of view. Moreover, the solution of inverse problems enables to observe or measure the processes in mass or temperature exchanger in real time and without influence of the system by insertion of measuring probes. It means without measuring inside the circuit of refrigerated liquid.

The organization of this thesis is following. First we need to summarize some basic knowledge in order to solve our problem. That is why we introduce mathematical tools in Chapter 2. Chapter 3 yields the introduction to the studied problem. In the next chapter we define terms like system, model, identifiability or observability. Then we have to mention internal links which are connected to our system. In Chapter 6, we describe the velocity profile of a liquid. In the following Chapter 7, we cover the rules for a heat transfer. The efficiency of the heat exchanger is dealt with in the following chapter. Until now, we have dealt with fundamental knowledge about heat exchangers. The original contribution of this thesis is contained in Chapters 9 and 10. The summary of types of compartment together with derivation of balance equation of given system is performed in Chapter 9. Then, we derive the solution under specific conditions in the Chapter 10. The final discussions are performed in Chapter 11 and the conclusion forms Chapter 12.

This master's thesis is based on the scientific stage of the author in BVT Technologies, a. s. During this, inverse problems of differential equations were solved and this master's thesis is a part of the internship. It was organized under the leadership of RNDr. Jan Krejčí, PhD. (BVT Technologies, a. s.) and Dr. Mark O'Connell (Probe Scientific, UK).

2 The Utilized Mathematical Tools

This section presents some basic mathematical background concerning the integral transforms, Green function and the Gordon-Klein equation.

2.1 The Integral Transform

We will use the Fourier transform and the Laplace transform for solving the Gordon-Klein equation which will be introduced later. Let us first call these integral transforms. (For more details see [2], [3], [4].)

Let $u(x,t)$ be an integrable complex-valued Lebesgue measurable function $u : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}$ and $t > 0$. The Fourier transform of $u(x,t)$ with respect to $x \in \mathbb{R}$ is denoted by $F\{u(x,t)\} = U(k,t)$ and we define it by the integral

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} u(x,t) dx.$$

where $k \in \mathbb{R}$ is called the transform variable. The inverse Fourier transform, denoted by $\mathcal{F}^{-1}\{U(k,t)\} = u(x,t)$ and we define it as

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{ikx} U(k,t) dk.$$

The other rellevant integral transform is the Laplace transform of $u(x,t)$ where $u(x,t)$ is any function defined in $a \le x \le b$ and $t > 0$, $a,\ b \in \mathbb{R}$ which satisfies

$$u(x,0) = \lim_{t\to 0^+} u(x,t), \qquad \text{uniformly} \tag{2.1}$$

$$|u(x,t)| < M\mathrm{e}^{c_0 t} \qquad \text{uniformly}, \tag{2.2}$$

where $t > 0,\ c_0 \in \mathbb{R},\ M > 0$. Then its Laplace transform with respect to t is denoted by $\mathcal{L}\{u(x,t)\} = L(x,s)$ and we define it by using the integral

$$\mathcal{L}\{u(x,t)\} = L(x,s) = \int_0^{\infty} \mathrm{e}^{-st} u(x,t) dt, \qquad \mathrm{R}\mathbf{e}\ s > 0,$$

where $s \in \mathbb{C}$ is called the transform variable. Under conditions (2.1) and (2.2) on $u(x,t)$, its transform $L(x,s)$ is an analytic function of s in the half plane $\mathrm{R}\mathbf{e}\ s > c$.

The inverse Laplace transform is denoted by $\mathcal{L}^{-1}\{u(x,s)\} = u(x,t)$ and defined by the complex integral

$$\mathcal{L}^{-1}\{L(x,s)\} = u(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{st} L(x,s) ds, \qquad c > 0.$$

2.2 The Green Function

We can describe many physical problems by second-order non-homogeneous differential equations with homogeneous boundary conditions or by second-order homogeneous equations with non-homogeneous boundary conditions. We can solve these problems by a method based on the Green function. (For more details see [4].)

We consider a non-homogeneous partial differential equation of the form

$$\mathbf{L}_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x}), \tag{2.3}$$

where $\mathbf{x} = (x, y, z)$ is a vector in three (or more) dimensions, $\mathbf{L}_{\mathbf{x}}$ is a linear partial differential operator in three or more independent variables with constant coefficients, and $u(\mathbf{x})$ and $f(\mathbf{x})$ are functions of three or more independent variables. The Green function $G(\mathbf{x}, \boldsymbol{\xi})$ of this problem satisfies the equation

$$\mathbf{L}_{\mathbf{x}} G(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \tag{2.4}$$

and represents the effect at the point $\mathbf{x}$ of the Dirac delta function of source at the point $\boldsymbol{\xi} = (\xi, \eta, \zeta)$. We multiply (2.4) by $f(\boldsymbol{\xi})$ and integrate over the volume V of the $\boldsymbol{\xi}$ space, so that $dV = d\xi d\eta d\zeta$, we obtain

$$\int_V \mathbf{L}_{\mathbf{x}} G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_V \delta(\mathbf{x} - \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} = f(\mathbf{x}). \tag{2.5}$$

We interchange the order of the operator $\mathbf{L}_{\mathbf{x}}$ and integral sign in (2.5) gives

$$\mathbf{L}_{\mathbf{x}} \left[\int_V G(\mathbf{x}, \xi) f(\boldsymbol{\xi}) d\boldsymbol{\xi} \right] = f(\mathbf{x}) \tag{2.6}$$

A simple comparison of (2.6) with (2.3) leads to the solution of (2.3) in the form

$$u(\mathbf{x}) = \int_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{2.7}$$

Clearly, (2.7) is valid for any infinite number of components of $\mathbf{x}$. Accordingly, we can apply the Green function method, in general, to any linear, constant coefficient, non-homogeneous partial differential equations in any number of independent variables.

Another way to approach this problem is by looking for the inverse operator $\mathbf{L}_{\mathbf{x}}^{-1}$. If it is possible to find $\mathbf{L}_{\mathbf{x}}^{-1}$, then the solution of (2.3) can be obtained as $u(\mathbf{x}) = \mathbf{L}_{\mathbf{x}}^{-1}(f(\mathbf{x}))$. It turns out that, in many important cases, it is possible, and the inverse operator can be expressed as an integral operator of the form

$$u(\mathbf{x}) = \mathbf{L}_{\mathbf{x}}^{-1}(f(\boldsymbol{\xi})) = \int_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{2.8}$$

The kernel $G(\mathbf{x}, \boldsymbol{\xi})$ is called the Green function which is, in fact, the characteristic of the operator $\mathbf{L}_{\mathbf{x}}^{-1}$ for any finite number of independent variables. In above mention the both expressions (2.7), (2.8) are the same.

2.3 The Gordon-Klein Equation

Let us deal with the one-dimensional inhomogeneous linear Gordon-Klein equation because it will be our goal to solve one at the end of this thesis. (The following ideas are taken from [4].) This equation has the form

$$u_{tt} - c^2 u_{xx} + d^2 u = p(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{2.9}$$

where c, d are constants and $u(x,t),\ p(x,t)$ are bounded and twotimes differentiable function. The initial and boundary conditions are given as

$$\begin{aligned} &u(x,0) = f(x) \quad \text{for } x \in \mathbb{R}, \\ &u_t(x,0) = g(x) \quad \text{for } x \in \mathbb{R}, \\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty,\ t > 0, \end{aligned}$$

where $f(x),\ g(x)$ are bounded functions satisfying

$$\begin{aligned} \lim_{|x|\to\infty} f(x) \to 0, \\ \lim_{|x|\to\infty} g(x) \to 0. \end{aligned}$$

At first we will solve simple task where $f(x) = 0 = g(x)$. We obtain

$$\begin{aligned} &u(x,0) = 0 = u_t(x,0) \quad \text{for } x \in \mathbb{R}, \\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty,\ t > 0. \end{aligned} \tag{2.10}$$

We need to set the Green function $G(x,t)$ for solving the problem. The Green function associated with the problem (2.9) satisfies the equation

$$G_{tt} - c^2 G_{xx} + d^2 G = \delta(x)\delta(t), \quad x \in \mathbb{R},\ t > 0. \tag{2.11}$$

Also, the same boundary and initial conditions as (2.10) are given for this equation (2.11). We apply the Fourier transform on the previous equation (2.11). Let $U(k,t) = \mathcal{F}\{G(x,t)\},\ k \in \mathbb{R}$ and $\delta(t)$ is Dirac delta. Then we obtain

$$\begin{aligned} U_{tt} - c^2(-ik)^2 U + d^2 U &= \frac{1}{\sqrt{2\pi}} \cdot \delta(t), && (2.12) \\ U(k,0) &= 0, && (2.13) \\ U_t(k,0) &= 0, && (2.14) \\ U(k,t) &\to 0 \qquad |k| \to \infty. \end{aligned}$$

By rearranging (2.12) we get

$$U_{tt} + \left(d^2 + c^2k^2\right) U = \frac{1}{\sqrt{2\pi}} \delta(t). \tag{2.15}$$

Now we can apply the Laplace transform to the equation (2.15). Let us suppose that $L(k,s) = \mathcal{L}\{U(k,t)\},\ s \in \mathbb{C}$ then we obtain

$$s^2 L(k,s) - sU(k,0) - U_t(k,0) + (d^2 + c^2k^2) L(k,s) = \frac{1}{\sqrt{2\pi}}. \tag{2.16}$$

Then we use the initial conditions (2.13) and (2.14) to (2.16) and we get

$$s^2 L(k,s) + (d^2 + c^2k^2) L(k,s) = \frac{1}{\sqrt{2\pi}}. \tag{2.17}$$

Hence,

$$L(k,s) = \frac{1}{s^2 + (d^2 + c^2k^2)} = \frac{1}{\sqrt{2\pi}} \frac{1}{s^2 + \omega^2}, \tag{2.18}$$

where $\omega^2 = d^2 + c^2k^2$. Now we determined the solution of the equation (2.17) to get the solution of (2.11) we have to apply the inverse Laplace and Fourier transform to the (2.18). At first we use the inverse Laplace transform $\mathcal{L}^{-1}\{L(k,s)\} = U(k,t)$.

$$U(k,t) = \mathcal{L}^{-1}\{L(k,s)\} = \frac{1}{\sqrt{2\pi}} \frac{sin(\omega t)}{\omega}. \tag{2.19}$$

We obtain the solution $G(x,t)$ of (2.11) by using inverse Fourier transform $G(x,t) = \mathcal{F}^{-1}\{U(k,t)\}$ to (2.19).

$$\begin{aligned} G(x,t) &= \mathcal{F}^{-1}\{U(k,t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega} \sin(\omega t) \mathrm{e}^{ikt} dk = \\ &= \frac{1}{2\pi c} \int_{-\infty}^{\infty} \left(\frac{d^2}{c^2} + k^2\right)^{-\frac{1}{2}} \sin\left(ct\sqrt{\frac{d^2}{c^2} + k^2}\right) \mathrm{e}^{ikt} dk = \\ &= \frac{1}{2c} J_0 \left[\frac{d}{c}\sqrt{c^2t^2 - x^2}\right] H(ct - |x|) = \\ &= \left[\begin{array}{cc} \frac{1}{2c} J_0 \left[\frac{d}{c}\sqrt{c^2t^2 - x^2}\right], & |x| < ct \\ 0, & |x| > ct \end{array} \right], \end{aligned} \tag{2.20}$$

where $J_0(x)$ is zero order Bessel function of the first kind and $H(x)$ is the Heaviside function. It is defined as

$$\begin{aligned} H(ct-|x|) &= 0 \qquad \text{for} \;\; ct-|x|<0, \\ H(ct-|x|) &= 1 \qquad \text{for} \;\; ct-|x|>0. \end{aligned}$$

The meaning of the Heaviside function is shown in the figure 2.1.

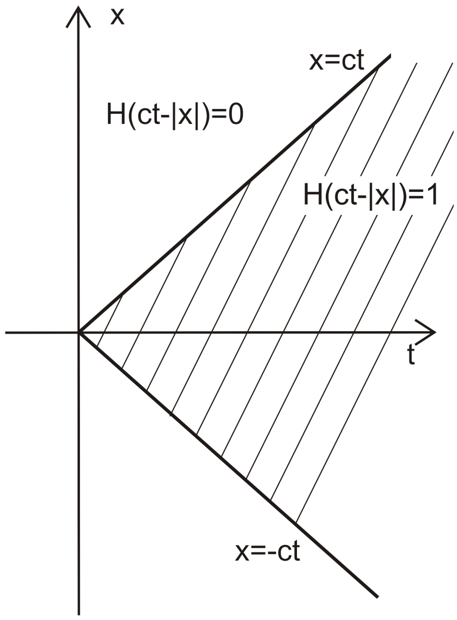

Fig. 2.1: The graph of the Heaviside function $H(ct-|x|)$

If the source is located at the point (ξ,τ), $\xi,\ \tau \in \mathbb{R}$ then the equation (2.11) is

$$G_{tt} - c^2 G_{xx} + d^2 G = \delta(x-\xi)\delta(t-\tau), \quad x \in \mathbb{R},\ t>0.$$

Further, Green function takes the form

$$G(x,t;\xi,\tau) = \begin{bmatrix} \frac{1}{2c} J_0 \left[\frac{d}{c}\sqrt{c^2(t-\tau)^2-(x-\xi)^2} \right], & |x-\xi|<c(t-\tau) \\ 0, & |x-\xi|>c(t-\tau) \end{bmatrix}. \quad (2.21)$$

The solution of the original Gordon-Klein equation (2.9) with the initial conditions

$$\begin{aligned} u(x,0) &= f(x) \qquad x \in \mathbb{R}, \\ u_t(x,0) &= g(x) \qquad x \in \mathbb{R}, \end{aligned}$$

can be expressed as

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_{-\infty}^{\infty} p(\xi,\tau) G(x,t,\xi,\tau) d\xi d\tau \\
&\quad + \int_{-\infty}^{\infty} \left[g(\xi) G(x,t,\xi,0) - f(\xi) G_\tau(x,t,\xi,0) \right] d\xi. \qquad (2.22)
\end{aligned}
$$

For the sake of better readability, we will solve separately the integrals in (2.22) to get the solution. The Green function G is equal to zero for $\tau > t$ it follows from (2.21). That is why we can write the upper limit in the double integral up to t. Further, it follows from (2.21) that $G \neq 0$ for $|x - \xi| < c(t - \tau)$ that means that $x - c(t-\tau) < \xi < x + c(t-\tau)$. So we can write

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} p(\xi,\tau) G(x,t,\xi,\tau) d\xi d\tau \\
&= \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} p(\xi,\tau) J_0 \left[\frac{d}{c} \{ c^2 (t-\tau)^2 - (x-\xi)^2 \}^{\frac{1}{2}} \right] d\xi. \qquad (2.23)
\end{aligned}
$$

Also we can derive the function $G(x,t;\xi,0)$ so then we obtain

$$
\int_{-\infty}^{\infty} g(\xi) G(x,t,\xi,0) d\xi = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) J_0 \left[\frac{d}{c} \{ c^2 t^2 - (x-\xi)^2 \}^{\frac{1}{2}} \right] d\xi. \qquad (2.24)
$$

Moreover, we need G_τ for the last term in (2.22). For this reason we rewrite (2.21) in terms of the Heaviside functions. So we get

$$
\begin{aligned}
G(x,t;\xi,\tau) &= \frac{1}{2c} J_0 \left[\frac{d}{c} \sqrt{c^2 (t-\tau)^2 - (x-\xi)^2} \right] \\
&\quad \times H\left[x + ct - (\xi + c\tau) \right] H\left[\xi + ct - (x + c\tau) \right]. \qquad (2.25)
\end{aligned}
$$

We obtain for G_τ

$$
\begin{aligned}
\left[G_\tau(x,t;\xi,\tau) \right]_{\tau=0} &= -\left(\frac{td}{2} \right) \frac{J_0' \left[\frac{d}{c} \sqrt{c^2 t^2 - (x-\xi)^2} \right]}{\sqrt{c^2 t^2 - (x-\xi)^2}} \\
&\quad \times H\left(x + ct - \xi \right) H\left(\xi + ct - x \right) \\
&\quad + \frac{1}{2} J_0 \left[\frac{d}{c} \sqrt{c^2 t^2 - (x-\xi)^2} \right] \\
&\quad \times \{ \delta(x + ct - \xi) H\left(\xi - x + ct \right) \\
&\quad + \delta(\xi - x + ct) H\left(x + ct - \xi \right) \}. \qquad (2.26)
\end{aligned}
$$

So, we get for the last term in (2.22) the form

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(\xi)G_\tau(x,t;\xi,0)d\xi &= -\left(\frac{td}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)d\xi \\
&+\frac{1}{2}\int_{-\infty}^{\infty} J_0\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right] \\
&\quad \times\{\delta(x+ct-\xi)H\left(\xi-x+ct\right)+ \\
&\quad\quad +\delta(\xi-x+ct)H\left(x+ct-\xi\right)\}f(\xi)d\xi.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(\xi)G_\tau(x,t;\xi,0)d\xi &= -\left(\frac{td}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)d\xi + \\
&+\frac{1}{2}\left[f(x-ct)+f(x+ct)\right].
\end{aligned} \tag{2.27}
$$

We finally receive the solution $u(x,t)$ by using the combination of (2.23), (2.24), (2.27).

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} J_0\left[\frac{d}{c}\sqrt{c^2(t-\tau)^2-(x-\xi)^2}^{\frac{1}{2}}\right] p(\xi,\tau)d\xi d\tau + \\
&+\frac{1}{2c}\int_{x-ct}^{x+ct} J_0\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right] g(\xi)d\xi - \\
&-\left(\frac{td}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)d\xi + \\
&+\frac{1}{2}\left[f(x-ct)+f(x+ct)\right].
\end{aligned}
$$

In the limit as $d \to 0$, this result is in agreement with the solution for the standard wave equation.

3 Introduction to the Problem

We deal with heat exchangers in this thesis. In order to say something more specific about heat exchangers we need to imagine them for the sake of clarity in the schematic way. Jacquez (see [5]) showed that we can imagine them as a system composed from some compartments whith a refrigerated liquid or a coolant. That is why we can describe it by the following relation

$$\begin{aligned}\frac{dx_i(t)}{dt} =&f_{i0}(x_1(t),\dots,x_n(t)) + \sum_{j=1,\ j\neq i}^{n} f_{ij}(x_1(t),\dots,x_n(t))\\ &- \sum_{j=1,\ j\neq i}^{n} f_{ji}(x_1(t),\dots,x_n(t)) - f_{0i}(x_1(t),\dots,x_n(t)), \quad i=1,2,\dots,n\,,\end{aligned} \tag{3.1}$$

where n is the number of compartments, x_i describes properties of a material in the compartment i (see [6]). In the case of a heat exchanger we talk about an amount of heat energy in an unit volume (see [7]). We denote an input to the compartment i as f_{ij} and an output from the compartment i as f_{ji}. f_{i0} describes the input of heat from the surrounding to the compartment i. Analogically f_{0i} is the output to the surrounding. If ($f_{0i} = 0,\ i = 1,2,\dots,n$) then we talk about closed system. In the other case the system is open. It is possible to imagine the equation (3.1) in the following scheme.

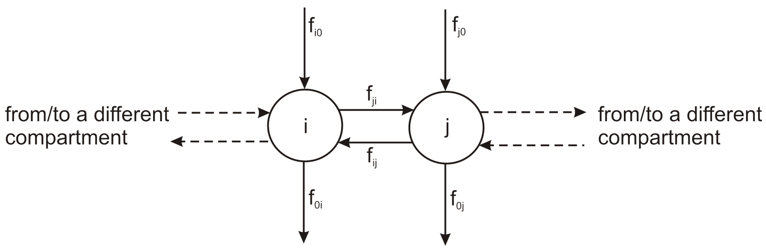

Fig. 3.1: Two compartments of general model described by (3.1)

How we can see from (3.1) the general model is described by a system of differential equations. However, in practice we usually use the linear model for which we can suppose that the flow is directly proportional to the amount of substance in the given compartment. We can express it as

$$\frac{dx_i(t)}{dt} = \sum_{j=1,\ j\neq i}^{n} k_{ij}x_j(t) - \sum_{j=1,\ j\neq i}^{n} k_{ji}x_i(t) - k_{0i}x_i(t) + u_i(t), \quad i=1,2,\dots,n, \tag{3.2}$$

where input flow f_{i0} is replaced by $u_i(t)$ which is the standard notation in the control theory of linear systems. We can display the equation (3.2) by changing the previous figure 3.1 for the following one.

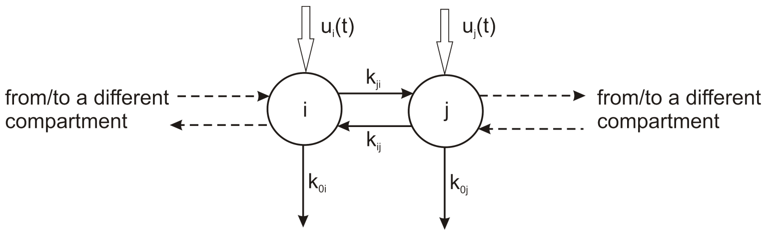

Fig. 3.2: Two compartments of linear model described by (3.2)

3.1 The Inverse problem

The aim is to define the basic idea of inverse problems.

We solve the problem as the inverse problem in this thesis because it can be very usefull to determine the behaviour of the whole system only from measurement of one compartment.

At the beginning we should tell something more about the inverse problem because it is not so common. First let us demonstrate the basic principle intuitively on the illustrative example. Then we will introduce the mathematical notation.

The basic idea of inverse problem is well demonstrated on the following example. Let us imagine that we have data about the spreading of muskrats in the Czech Republic. We can display this data in the map in the figure 3.3.

The usual approach would be to determine the area and how many of muskrats will be in a next year for example. The spreading of muskrats governs by diffusion model. It is possible to solve similar problem like in chapter 2.3. Knowing the initial and boundary conditions we can forecast the development in future. However, the inverse problem has the reverse direction. The basic task of inverse problem lies in questions

- When was the first muskrats introduced in nature.
- Where was the first muskrats introduced in nature.
- How many was the first muskrats introduced in nature.

There we can also see the difficulty of the task. The problem is in the moment when we have to decide to stop the solution. The mathematical model does not have any problem with one half of a muskrat running somewhere in the wood. But from the practical point of view we know that this is absolute nonsense. If we cross

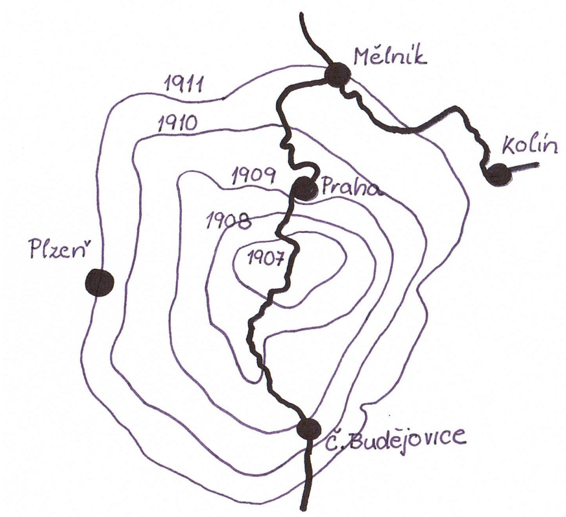

Fig. 3.3: The spread of muskrats in the Czech Republic (see [8])

the line of reality the model starts to behave very unstable. The inverse task are not property possesed in general.

Now we introduce the mathematical background of the inverse problem. (For more details see [9].)

We will deal with boundary value problems for linear differential equations. We introduce the studied linear differential equation with boundary conditions.

Let $\langle \alpha, \beta \rangle \subset \mathbb{R}$ be a compact interval and let $M_{ij},\ N_{ij},\ c_i \in K$ for $i = 1, 2, \dots, r$, $j = 1, 2, \dots, n$, where K is $\mathbb{R}$ or $\mathbb{C}$. Further, let the functions $h, a_j : \langle \alpha, \beta \rangle \to K$ be continuous for $j = 0, 1, \dots, n$, $a_0(t) \neq 0$ for $t \in \langle \alpha, \beta \rangle$ and u let be a real function which has all derivatives up to the order n. The task to find the solution u of the equation

$$a_0(t) \cdot u^{(n)}(t) + a_1(t) \cdot u^{(n-1)}(t) + \cdots + a_n(t) \cdot u(t) = h(t) \tag{3.3}$$

satisfying the boundary conditions

$$\sum_{j=1}^{n} M_{ij} u^{(j-1)}(\alpha) + \sum_{j=1}^{n} N_{ij} u^{(j-1)}(\beta) = c_i, \qquad i = 1, 2, \dots, r \tag{3.4}$$

is called the boundary value problem. For a better readability, we can write the boundary conditions (3.4) in the vector form. Using the notation

$$
\begin{aligned}
&\mathbf{u}(t) = \left[u(t), \dot{u}(t), \dots, u^{(n-1)}(t)\right]^T \in K^n, \\
&\mathbf{M} = (M_{ij}), \ \mathbf{N} = (N_{ij}), \\
&\mathbf{c} = [c_1, c_2, \dots, c_r]^T \in K^r,
\end{aligned}
$$

where $\mathbf{M}, \ \mathbf{N}$ are matrices of the type $r \times n$ and the upper index T denotes the transpose matrix, we can write the boundary conditions as

$$
\mathbf{Mu}(\alpha) + \mathbf{Nu}(\beta) = \mathbf{c}. \tag{3.5}
$$

Now let $C^{(0)} = C^{(0)} (\langle \alpha, \beta \rangle \to K)$ be the space of continuous functions which are mapping the interval $\langle \alpha, \beta \rangle$ into K and let $C^{(n)} = C^{(n)} (\langle \alpha, \beta \rangle \to K)$ be the space of functions from $C^{(0)}$ having continuous derivatives up to the order n.

Left-hand sides of (3.3) and (3.5) determine the operator $\mathrm{L} : C^{(n)} \to C^{(0)} \times K^r$. This operator L maps the function u into the couple involving the function on the left-hand side of (3.3) and the vector on the left-hand side of (3.5).

Our aim is to describe the set of values of L and answer the question for which couples $(h, \mathbf{c})$ the problem (3.3), (3.5) has a solution. In addition to the previous assumptions we suppose that the function a_j has continuous derivatives up to the order $n - j$.

We introduce the operators $l, \ l^* : C^{(n)} \to C^{(0)}$ by

$$
\begin{aligned}
&l(u) = a_0(t)u^{(n)} + a_1(t)u^{(n-1)} + \dots + a_n(t)u, \\
&l^*(v) = (-1)^n \left(\overline{a}_0(t)v\right)^{(n)} + (-1)^{n-1} \left(\overline{a}_1(t)v\right)^{(n-1)} + \dots + \overline{a}_n(t)v,
\end{aligned} \tag{3.6}
$$

where l^* is the adjoin operator of l, $\overline{a}_j$, $j = 0, 1, \dots, n$, are the coefficients which are complex conjugate functions of a_j and $v \in C^{(n)}$. For the same reason as in the case of u we write

$$
\mathbf{v}(t) = \left[v(t), \dot{v}(t), \dots, v^{(n-1)}(t)\right]^T.
$$

Further, we introduce the regular matrices $\mathbf{\Theta}(t) = (\Theta_{ik}(t))$, $\mathbf{\Theta}^*(t) = (\Theta^*_{ik}(t))$ where $\Theta^*_{ik}(t) = \overline{\Theta}_{ki}(t)$, $\Theta_{ik}(t) = 0$ for $i + k > n + 1$, $\Theta_{ik}(t) = (-1)^{i-1} a_0(t)$ for $i + k = n + 1$, $t \in \langle \alpha, \beta \rangle$.

Now let $v \in C^{(n)}$, $\mathbf{w} \in K^r$ and let

$$
l^*(v) = 0, \tag{3.7}
$$

$$
\mathbf{M}^*\mathbf{w} - \mathbf{\Theta}^*(\alpha)\mathbf{v}(\alpha) = 0, \quad \mathbf{N}^*\mathbf{w} + \mathbf{\Theta}^*(\beta)\mathbf{v}(\beta) = 0, \tag{3.8}
$$

where $\mathbf{M}^*$, $\mathbf{N}^*$ are adjoint matrices to $\mathbf{M},\ \mathbf{N}$. Then (3.7), (3.8) are called the adjoint problem to (3.3), (3.5).

If (3.3), (3.5) has a solution then

$$\int_{\alpha}^{\beta} h(t)\overline{v}(t)dt + (\mathbf{c}, \mathbf{w}) = 0, \tag{3.9}$$

where $(\mathbf{c}, \mathbf{w})$ is the inner product of $\mathbf{c},\ \mathbf{w}$ which is defined as

$$(\mathbf{c}, \mathbf{w}) = \sum_{i=1}^{r} c_i w_i.$$

Conversely, if (3.9) hold for any solution $(\mathbf{v}, \mathbf{w})$, where $\mathbf{v} \in C^{(n)}$ and $\mathbf{w} \in K^r$ are satisfying (3.7), (3.8) then (3.3), (3.5) has a solution.

As the formulation of the inverse problem we understand the task to find the operator L which transforms the solution to the differential equation and its initial and boundary conditions.

4 The Properties of Model and Parameters

It is necessary to answer the question of the behaviour of the system for the given input and initial parameters which in turn eliminates possible errors in measurement. Therefore, it is easier to decompose the problem to a few degrees. First, we prepare a model for the system. To do this, we need to know a structure of the system. Recall that some issues related to this problem we can answer without unique model. The structure of the system is determined by all the unknown parameters, therefore it is necessary identified them.

Modelling and simulation enables us to understand the basic and essential characteristics of the system and thus be better described. Therefore, it is necessary to define the basic characteristics of the parameters.

Input parameters of the cooling circuit, i.e. the input flow and its temperature, can be set without major restrictions accurately, thus they can be controled. Output parameters, i.e., the output of flow and its temperature are not controllable because they are affected by the presence of the circuit of the refrigerated liquid. However, there are no fundamental limitations to measure output parameters. We can observe them quantitatively. The aim is to determine under what conditions the input and output of cooled liquid is measurable. Further, under which conditions it is possible to determine the temperature profile inside the heat exchanger based on the controlling of the input in the circuit of coolant and measuring the output temperature. This leads us to the question of controllability, observability and identifiability of parameters in the circuit of refrigerated liquid. It is clear that if they are not controllable parameters they can be still observed. The concepts of controllability, observability and identifiability will be accurately introduced later.

4.1 The Difference between System, Model and Experiment

It is necessary to differentiate the terms system, model and experiment.

The system is the connection of the compounds which are related and depended one each other and which creates the integrated unit. As a model we understand the simplification of a system which contains its significant properties. These properties can be express by mathematical relations and parameters.

Then we conduct experiments on our system. The experiment is methodical sequence of attempts and mistakes which are done with the purpose to verify or determine the validity of hypothesis (see [10]). We can decide if the model is precise enough by comparing of a prediction of model and results from the experiment.

We have to identify parameters of model to be able to apply it. However, this is not always possible. That means that it can happen that we have great model but the parameters are not identifiable. That can be illustrated as

$$
\begin{aligned}
f\left(x_1, \frac{r_1}{r_2}, r_3\right) &= 0, \\
f\left(x_2, \frac{r_1}{r_2}, r_3\right) &= 0, \\
&\vdots \\
f\left(x_n, \frac{r_1}{r_2}, r_3\right) &= 0,
\end{aligned}
\tag{4.1}
$$

where the function f is a function of three variables one controllable independent variable and two variables which serves as parameters. We can make n measurement with different values x_n but it is impossible to find $r_1,\ r_2,\ r_3$ from these equations. Even if we will work with precise value it can happen that some parameters are not observable nor identifiable. The equation $f(x, \frac{r_1}{r_2}, r_3) = 0$, describes the model of system, which depends on the independent variable x. For n values of x ($x = x_i$ where $i = 1, 2, \ldots, n$) we obtain n equations for two unknown $\frac{r_1}{r_2}$ and r_3. But still we are not able to set r_1 nor r_2 because they are in the fraction. In practice we obtain the other uncertainty and it is the precision of measuring and statistical behaviour of measured data.

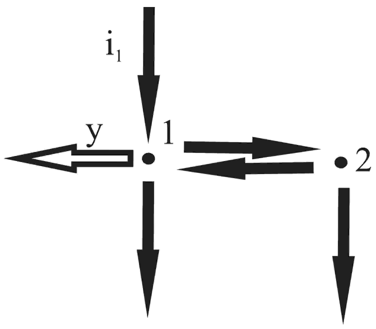

Fig. 4.1: The example of the unidentifiable system

We have to know the structure of the system to compile the appropriate model. In the figure 4.1, we can see the structure of the system which has the input i_1 in the point 1 and we also measure the output y there. However, there are some losses to the surroundings through the point 2. So, we are not able to tell anything about the system from the measured data.

It follows from the previous example that it is necessary to define the structure of the system. That is why we introduce the term of the structural identifiability.

Chis distinguish the structural and the practical identifiability where the structural identifiability denotes the theoretical property of the structure of a model which depends only on the dynamic of the system (see [11]). The practical identifiability is connected with experimental data and experimental noise.

4.2 The Identifiability, Observability and Controllability

Model is identifiable if we can measure each compartment. This means, that we can define all of its parameters (see [5]). Parameter is globally identifiable if only one its value is expressed in accordance with results of the experiment after substituting into the model. If this property is met only at some neighbourhood of a parameter value, we call this parameter locally identifiable (see [12]).

In practice, it is necessary not only to measure the system outputs but control them very often, too. System controllability is also necessary to stabilize it. System is controllable if it is impossible to separate the state variables into two groups such that one of them is not affected by using a second group of state variables or input parameters. It is obvious that if parameters are not controllable they may still be observable (see [10]).

Sometimes observability is known like measurability. It determines if it is possible from the measured values of the output to make any conclusions about the system or not. So, if we are able to determine the behaviour of the input system from its outputs.

Modelling and simulation enables us to understand the basic and essential characteristics of the system, so it can be better described. It is therefore necessary to define the basic properties of the parameters. The input parameters of the coolant circuit, i.e. the input flow rate and temperature, we can, without any major limitation set, then control. Input parameters of the cooling circuit are controllable. Output parameters, i.e. the output flow rate and temperature, are not controllable because they are influenced by the presence of the cooling circuit. However, there is no principled limit to measured output parameters. Thus, we can quantitatively observe them. Output parameters are observable. The aim is to determine under what conditions the input and output of refrigerated liquid is measurable. Alternatively, under what conditions it is possible to determine the temperature profile of heat exchanger on the basis of coolant access control of input coolant and measurement of temperature of output dialysate. This leads us to the question of controllability, observability and identifiability of parameters of the cooled circuit.

5 A Priori Internal Links

In our case we talk about system which is inseparably connected with the flow of liquids. That is why we need to set some physical conditions for our model. Let us define the symbols and notations we will use. Let $\mathbf{x} = (x_1, \ldots, x_d)$ be any point occupied by the fluid where $d \in \mathbb{N}$ is the dimension of a bounded domain $\Omega \subseteq \mathbb{R}^d$ occupied by the fluid. (For more details see [13].)

5.1 The Transport Theorem

For describing the change of the physical quantity $F(\mathbf{x}, t)$ we define the total amount of the quantity given by the differentiable function F contained in the control volume $V \subseteq \Omega$ at time $t > 0$ as

$$\mathfrak{F}(t) = \int_V F(\mathbf{x}, t)\, d\mathbf{x}.$$

We can study the time change of the $\mathfrak{F}(t)$

$$\frac{d\mathfrak{F}(t)}{dt} = \frac{\partial}{\partial t} \int_V F(\mathbf{x}, t)\, d\mathbf{x},$$

$$\frac{d\mathfrak{F}(t)}{dt} = \int_V \left[\frac{\partial F}{\partial t}(\mathbf{x}, t) + \operatorname{div}(F\mathbf{v})(\mathbf{x}, t) \right] d\mathbf{x}, \tag{5.1}$$

where $\mathbf{v}$ is the velocity of the flow. The equation (5.1) is the so called transport theorem. By using the Green's theorem we get

$$\frac{d\mathfrak{F}(t)}{dt} = \int_V \frac{\partial F}{\partial t}(\mathbf{x}, t)\, d\mathbf{x} + \int_{\partial V} F(\mathbf{x}, t)\, \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})\, d\mathbf{x}, \tag{5.2}$$

where $\mathbf{n}(\mathbf{x})$ is a unit outer normal to ∂V at the point $\mathbf{x}$.

The equation (5.2) expresses that the change of $\mathfrak{F}(t)$ consists of its explicit dependance on the time and flow of $\mathfrak{F}(t)$ through the walls of studied volume. In many cases the $\mathfrak{F}(t)$ is conserved. For example, total mass, total volume, total energy of system.

5.2 The Continuity Equation

Now, we derive the continuity equation by assuming the incompressibility of a liquid. Let us suppose that

$$\mathfrak{F}(t) = \int_V 1 \cdot d\mathbf{x}.$$

Hence,

$$\frac{d\mathfrak{F}(t)}{dt} = \frac{\partial}{\partial t}\int_V 1 \cdot d\mathbf{x} + \frac{\partial}{\partial \mathbf{x}}\int_V 1 \cdot \frac{\partial \mathbf{x}}{\partial t} d\mathbf{x} = \text{div } \mathbf{v} = 0.$$

Hence, we can write the incompressibility condition as

$$\text{div } \mathbf{v} = 0.$$

In our case of heat exchangers we have a closed system with respect to the mass. It means that no liquid can leak anywhere in the system. We can imagine it as in the figure 5.1 where the bold lines denotes the impermeable walls.

Fig. 5.1: The closed system with respect to the conservation of the total volume

So, we obtain the continuity equation in the form

$$Q_{SI} + Q_{DI} = Q_{SO} + Q_{DO}, \tag{5.3}$$

where Q is the flow of the liquid and I means the flow on the input, O is for the output. S means that the liquid is refrigerated and D stands for a coolant. The equation (5.3) describes that liquid cannot accumulate anywhere inside the system, it cannot leak anywhere and it cannot be created inside the system. Hence, it follows that the sum of the flows at the input of the system has to be equal to the sum of the flows at the output of the system.

5.3 The Conservation of Mass

We would like to introduce the continuity equation. However, first we have to tell something about the conservation of mass because the continuity equation follows from the first physical law about fluid motion which is called the law of conservation of mass. We obtain it from the (5.1) if we take $F(\mathbf{x}, t) = \rho(\mathbf{x}, t)$. We get

$$\mathfrak{F}(t) = \int_V \rho(\mathbf{x}, t) d\mathbf{x},$$

where $V \subseteq \Omega$ is a control volume and $\rho(\mathbf{x}, t)$ is a density of a liquid depending on position $\mathbf{x}$ and time t. Now the $\mathfrak{F}(t)$ is the total amount of mass in the volume V which cannot be changed. Thus,

$$\frac{d}{dt}\mathfrak{F}(t) = 0. \tag{5.4}$$

The equation (5.4) can be formulated as: The mass of a piece of fluid formed by the same particles at any time instant is constant in time. In other words the mass cannot be created nor destroyed.
By using the Transport theorem (5.1) we can formulate the law of conservation of mass in the following form

$$\int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v})\right] d\mathbf{x} = 0. \tag{5.5}$$

We can express (5.5) in differential form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \qquad \text{in } \Omega \times (0, T). \tag{5.6}$$

Thus,

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div}(\mathbf{v}) + \mathbf{v}\ \mathbf{grad}(\rho) = 0 \qquad \text{in } \Omega \times (0, T).$$

So, we receive

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\ \mathbf{grad}(\rho) = 0 \qquad \text{in } \Omega \times (0, T).$$

The law of conservation of mass is illustrated in the following figure.

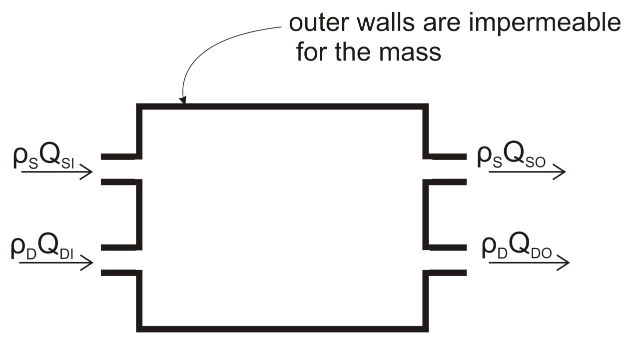

Fig. 5.2: The closed system with respect to the mass

Hence,

$$\rho_S Q_{SI} + \rho_D Q_{DI} = \rho_S Q_{SO} + \rho_D Q_{DO},$$

where ρ_S is the density of the cooled liquid and ρ_D stands for coolant.

5.4 The Law of Conservation of Energy

The law of conservation of energy says that the energy cannot be created nor destroyed but it can be transformed. By assuming this condition we can formulate

$$dU\left(V\right) = dW\left(V\right) + dJ_Q\left(V\right) \tag{5.7}$$

where W is the power, J_Q is the rate of heat addition and dU is the internal energy. The equation (5.7) says that we can changed the energy of a system by performing some work or by adding/removing some heat energy.

We can express the law of conservation of energy by using the transport theorem as

$$\mathfrak{F}(t) = \int_V u\left(\mathbf{x}, t\right) d\mathbf{x},$$

where $u(\mathbf{x}, t)$ is the density of the internal energy and

$$\frac{d\mathfrak{F}(t)}{dt} = 0.$$

By applying the law of conservation of energy for heat exchangers we get the following form

$$Q_{SI}T_{SI}c_S + Q_{DI}T_{DI}c_D = Q_{SO}T_{SO}c_S + Q_{DO}T_{DO}c_D \tag{5.8}$$

where c is a specific heat capacity, T denotes temperature and the meaning of subscripts is the same like above. We can display the equation (5.8) as the following figure.

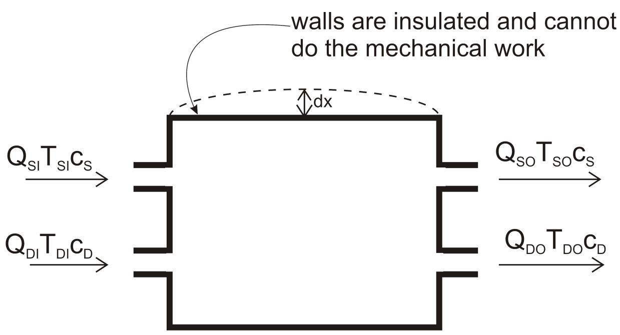

Fig. 5.3: The closed system with respect to the energy

The bold lines have to satisfy two assumptions. At first they have to be a thermal insulator. Thus, the heat cannot pass through the walls in any direction. The second condition is that the material has to be perfectly mechanically fixed because any motion of the wall of dx will cause the work $p \cdot S \cdot dx$ where p is pressure and S is the surface. If p or S is big then very small dx can cause a big change of the internal energy. The only change of internal energy is given under the previous assumptions by the ingoing and the leaving heat. Thus, the equation (5.8) is satisfied.

6 The Velocity Profile

In our task we deal with a flow of a liquid. That is why we should talk about velocity profile. It will be shown that it can influence our problem in a very important way. We have to formulate some hydrodynamic properties of the liquid because they directly affect the process of heat exchange.

From the experience it is known that movement of the liquid leads to the dissipation of energy. It is caused by an interaction between different parts of the liquid with a different velocity. Thus, the faster part of liquid affects the slower part of the liquid. We can say that each molecule of the liquid transmits a certain amount of mass, momentum and energy. That is what leads to the diffusion, internal friction and heat conduction. It is obvious that these processes are irreversible from the thermodynamic point of view (see [14]).

Let us discuss some of the physical properties of the liquid which affect the hydrodynamic behaviour. These include the density ρ which is mass per volume consequently $\rho = \frac{m}{V}$ and the unit is $\frac{kg}{m^3}$. Another feature is the stress tensor τ_{ij} with the unit $N \cdot m^{-2}$ which characterizes the force action inside the moving liquid. It consists of two parts. The first one describes the behaviour of perfect liquid which is denoted by 1 in the equation (6.1) and the second denotes stress caused by the viscosity of liquid is denoted by 2 in the equation (6.1). That means that we can rewrite τ_{ij} as

$$\tau_{ij} = \underbrace{-\delta_{ij} p}_{1} + \underbrace{\tau'_{ij}}_{2}, \tag{6.1}$$

where $p \geq 0$ is a pressure with unit Pa. We can demonstrate the meaning of τ'_{ij} in (6.1) by using the following example. Let us imagine a system which consists of two parallel straight plates and a liquid between them which is viscous. Let us suppose that the distance h between these two plates is constant. Then we can imagine that in some moment the upper plate starts to move with a constant velocity $\mathbf{v}_0$ and the bottom plate stays stationary. After that we can observe that individual layers of the liquid start to move. That is caused by τ'_{ij}. The whole situation is schematically represented in the figure 6.1.

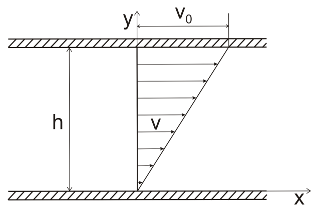

Fig. 6.1: The demonstration of τ'_{ij}

Then there can occur two possibilities. If the liquid would be not viscous then the upper plate would just slightly move and nothing else would happen. However, in our case the liquid is viscous so the layers of liquid will start to move also and it is caused by viscosity. The final velocity profile will be linear like we can see in the figure 6.1. We call this way of a flow the laminar flow or the streamline flow. That means that a liquid flows in parallel layers with no disruption between the layers. If there would appear some disruption we call this kind of flow a turbulent flow or just turbulence. The impact of the type of flow on the velocity profile is shown in the figure 6.2.

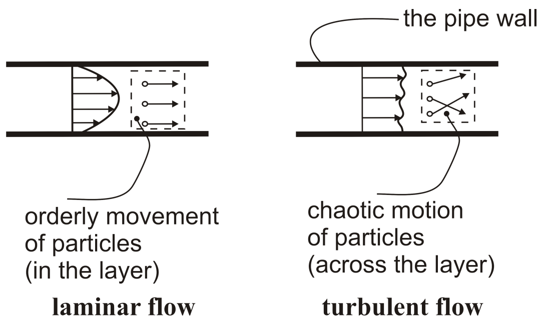

Fig. 6.2: The difference between laminar and turbulent flow

Let us say something more specific about viscosity. We distinguish between two kinds of viscosity, the kinematic and the dynamic one. The dynamic viscosity μ gives us the relation between stress tensor τ'_{ij} caused by viscosity and change of velocity $\mathbf{v}$ depending on position. Thus the unit is $\frac{kg}{m \cdot s}$. With respect to the coordinate system in the figure 6.1 we can write

$$\tau'_{xy} = \mu \frac{\partial \mathbf{v}}{\partial y}.$$

We will get the kinematic viscosity ν from dynamic viscosity μ by dividing by density ρ. Hence,

$$\nu = \frac{\mu}{\rho},$$

where kinematic viscosity has the unit $\frac{m^2}{s}$. Both dynamic and kinematic viscosity depend on the type of the fluid used and on the temperature.

6.1 The Law of Similarity

The motion of liquid is described by Navier-Stokes equations. Many questions connected with their solutions are still not solved. This is why the question of similarity is very important in hydrodynamic because the routine is that experiment is performed on the model of a different size. This model has to preserve the dimensions ratio of the original one. It is not enough for the successful experiment to guarantee only the geometrical similarity of shape but also all forces applied in system have to be in the same ratio. Hence, the system has to have the same kind of flow, either laminar or turbulent. If all previous assumptions are satisfied we can get the behaviour of the real system by using appropriate scaling. Then we can determine the real hydrodynamic behaviour of our system. The necessity of the law of similarity follows from the possibility to derive a lot of information about our system from model. For this reason we set the characteristic number which is the dimensionless number which describe the character of the flow. There are many definitions of characteristic numbers like Richardson, Freude, Stokes, Strouhal number etc. The most important of them are Reynolds, Nusselt and Prandtl number.

Reynolds number Re is one of the most famous. Its definition for a flow in a pipe or a tube is the following

$$Re = \frac{\mathbf{v}_M \phi_H}{\nu},$$

where $\mathbf{v}_M$ denotes the mean velocity of the liquid, ν is the kinematic viscosity and ϕ_H is the hydraulic diameter of the pipe. The hydraulic diameter is defined as

$$\phi_H = \frac{4A}{P},$$

where A is the cross sectional area and P is the wetted perimeter of the cross-section. It is possible to determinate the type of the flow from the value of Reynolds number. The critical value is $Re_{crit} = 2320$. The laminar flow occur for $Re \leq Re_{crit}$. Otherwise, thus $Re > Re_{crit}$ we talk about turbulent flow (see [15]).

The flow is influenced by the surface of a tube or a pipe. We can express this relation by using a friction coefficient λ which depends on the type of flow and that is why we distinguish two kinds of formulas for λ. For a laminar flow we define

$$\lambda = \frac{A}{Re},$$

where A is the cross sectional area and Re is Reynolds number. We write for a turbulent flow

$$\lambda = \frac{0.3614}{\sqrt[4]{Re}}.$$

The other important characteristic number is Nusselt number Nu which is defined as

$$Nu = \frac{\text{convective heat transfer}}{\text{conductive heat transfer}} = \frac{\alpha L}{\lambda},$$

where α is the thermal conductivity of the fluid with the unit $\frac{W}{m^2K}$, λ is the convective heat transfer coefficient of the fluid and L is the characteristic length. Nusselt number describes the ratio of convective to conductive heat transfer across the boundary (see [16]).

The next one is the Prandtl number Pr which gives the ratio between kinematic viscosity and thermal diffusivity.

$$Pr = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\nu}{a} = \frac{c_p\mu}{\lambda},$$

where $a = \frac{\lambda}{\rho c_p}$ is a thermal diffusivity with unit $\frac{m^2}{s}$ and c_p is the specific heat with unit $\frac{J}{kgK}$, λ is a thermal conductivity, μ is a dynamic viscosity, ν is a kinematic viscosity.

It is also possible to set the relation between them. For example we can express the Nusselt number as

$$Nu = kRe^n Pr^m,$$

where $k, m, n \in \mathbb{R}$ are constants empirically given by experimental data.

7 The Heat Transfer

The heat transfer may be done in three possible ways. The conduction can only take place within an object or material, or between two objects that are in direct or indirect contact with each other. We can imagine that as the situation when we put part of an iron rod to a fire and we can observe that after some time the part which is not in the fire gets hot. The convection is possible to occur when the temperature in the whole volume is not the same and it stabilizes by convection into the steady-state. Convection can be demonstrated by placing a heat source at a side of a glass full of a liquid, and observing the changes in temperature in the glass caused by the warmer fluid moving into cooler areas. The last possibility is the thermal radiation. Thermal radiation is generated when the energy from the movement of charged particles within atoms is converted to electromagnetic radiation. Infra-red radiation from a common household radiator or electric heater is an example of thermal radiation, as is the heat emitted by an operating incandescent light bulb.

7.1 The Heat Exchanger

The heat exchanger deals with convection and conduction at the same time which of them is the main one depends on the properties of liquid and the kind of flow. That is why it is necessary to consider all hydrodynamic properties. (The following ideas are taken from [17].)

The heat transfer is materially influenced by the hydrodynamic process if the liquid is very close to a solid wall. It follows from the velocity profile that particles which flow very close to the surface of a solid object have almost zero velocity. That means that adhesion occurs together with the boundary layer. The amount of transformed heat depends on the type of the flow. The thermal boundary layer is significantly different for the laminar flow and for the turbulent flow. The thermal boundary layer occurs in the immediate vicinity of the solid wall where are large changes of temperature. They are caused by a heat transfer from the wall to the liquid or in the opposite direction. The temperature progress in the vicinity of solid wall is displayed in the figure 7.1.

We can describe the heat transfer by Newton's Law of Cooling which states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings.

$$\begin{aligned} J_Q &= \alpha S(T_M - T_W) \quad \dots \text{ cooling,} \\ J_Q &= \alpha S(T_W - T_M) \quad \dots \text{ heating.} \end{aligned} \tag{7.1}$$

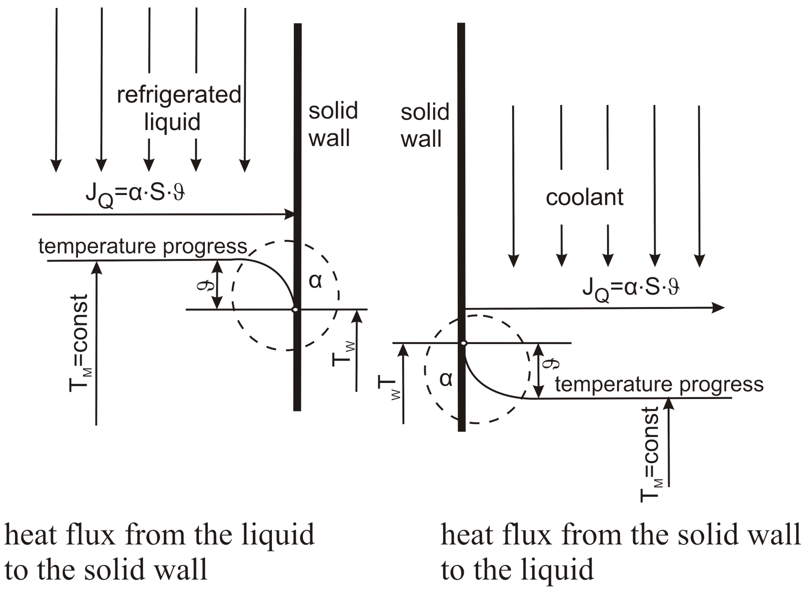

Fig. 7.1: The heat transfer in the thermal boundary layer

where J_Q denotes the heat flow rate with the unit W, S is the surface of the contact area and the unit is m^2 and finally T denotes the temperature with the unit K then T_W is the temperature of a wall and T_M is the mean temperature of a liquid, α is the individual convection heat transfer coefficient and its unit is $\frac{W}{m^2 \cdot K}$. It denotes the properties of the interface of two different substances (see [18]).

The individual convection heat transfer coefficient gives us the amount of heat energy which is able to penetrate across the wall with surface $1m^2$ if the temperature difference is $1K$ thus $\Delta T = |T_M - T_W| = 1K$. The coefficient α depends on the physical properties of the liquid, on the velocity of the flow and on the shape surface of the wall on the both sides. That is why is its value determined experimentally in dependence on different factors which influence the amount of shared heat energy. Hence, $\alpha = f(v, T_W, T_M, \lambda, c, \rho, \eta, \Phi, w, h, l)$ where w, h, l denotes the dimensions of a wall, Φ characterize the shape of the wall, v is the velocity of a liquid, λ is the thermal conductivity of the wall with unit $\frac{W}{m \cdot K}$ and it express the ability of a material to conduct heat, c is the heat capacity per particle with unit $\frac{J}{kg \cdot K}$, ρ is the density of a liquid and η is its dynamic viscosity with unit $\frac{N \cdot s}{m^2}$.

The liquid creates the boundary layer on the wall where the velocity of the flow is close to zero. That is why it is possible to consider that the heat energy is shared only by conduction. This idea is expressed by Fourier's Law of thermal conduction as

$$j_Q = -\lambda \text{grad}(T), \tag{7.2}$$

where j_Q denotes the heat flux. The minus on the right side express that the heat flux has always the opposite direction then the temperature gradient. In other words, the heat flux goes from the position with higher temperature to the position with lower temperature. It is possible to express an equation of heat flux of given system by assuming the Fourier's Law as

$$j_Q = -\lambda\left(\frac{\partial T}{\partial n}\right),$$

where n characterize the direction of the normal vector to the solid wall and $-\left(\frac{\partial T}{\partial n}\right)$ determinates the temperature gradient.

Let us suppose that the temperature of the refrigerated liquid T_S is higher than the temperature of the cooler T_D. The heat transfer coefficient u for the whole exchanger is its characteristic parameter. Generally, this coefficient it is very difficult to determine. It expresses the thermal transmittance of the wall that separates the two environments consist of the gaseous or liquid substances (see [18]). Its meaning follows from the figure 7.2.

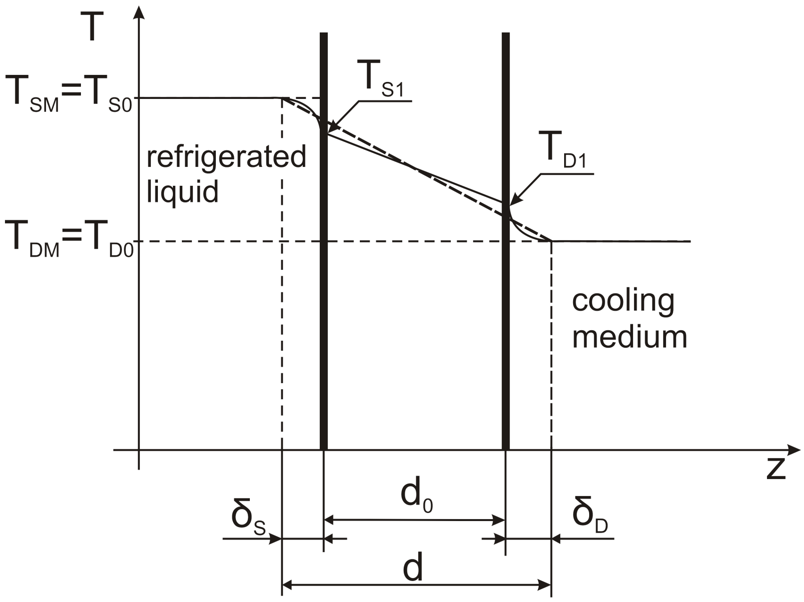

Fig. 7.2: The temperature profile in the active area of the heat exchanger

The temperature T_{SM} denotes the temperature in the area of the refrigerated liquid which is not influenced by the contact area. The contact area causes the temperature of the refrigerated liquid to fall to the temperature T_{S1} in the boundary layer with thickness δ_S. The same situation is in the compartment of the coolant where the thickness of the boundary layer δ_D can be different depending on the type of the liquid. Under the assumption of the steady state, thus $\frac{\partial T}{\partial t} = 0$ is the drop in temperature inside the wall defined by the thermal conductivity of its material. Then the heat flux under the steady state condition is characterized as

$$j_Q = -\lambda \frac{T_{D1} - T_{S1}}{d_0},$$

where d_0 is the thickness of the contact area. The substance with temperature T_{SM} enters the boundary layer with thickness δ_S. This is caused mainly by hydrodynamic as the corollary of flow. The thickness of the boundary layer is between $1 - 100\mu m$ and it depends on the type of the flow, viscosity of the liquid and also on the specific hydrodynamic properties of given liquid. The heat energy has to be transferred from the boundary layer of the refrigerated liquid to the wall of the heat exchanger. In the wall, the heat energy is passing according to the Fourier's Law (7.2). Then again from the wall to the boundary layer of the coolant and then to the whole volume of the coolant where the temperature is T_{DM}. Hence, it is possible to set the total heat flux as

$$j_Q = \frac{\Delta T}{R}, \tag{7.3}$$

where R denotes the total heat resistance and $\Delta T = |T_{SM} - T_{DM}|$. The resistance of given system can be expressed as

$$R = \frac{1}{\alpha_S} + \frac{\lambda_W}{d_0} + \frac{1}{\alpha_D},$$

where α_S is an individual convection heat transfer coefficient of the refrigerated liquid and α_D is for the coolant, λ_W is the thermal conductivity of the wall with thickness d_0. By substitution to (7.3) we obtain

$$j_Q = \frac{T_{SM} - T_{DM}}{\frac{1}{\alpha_S} + \frac{\lambda_W}{d_0} + \frac{1}{\alpha_D}}.$$

For the sake of better readability, let us substitute the total resistance with the overall heat transfer coefficient u

$$u = \frac{1}{R},$$

where its unit is $\frac{W}{m^2 K}$. Then we obtain from (7.3) the following form

$$j_Q = u \Delta T_S,$$

where ΔT_S is the mean difference of temperatures in the heat exchanger which is changed along the contact area and u is the overall heat transfer coefficient. Now it is possible to express its dependence as

$$u = \frac{j_Q}{\Delta T_S}.$$

Let us consider two possible cases for a demonstration of the effect of hydrodynamic properties. The parallel flow exchanger is on the left and the antiparallel is on the right in the figure 7.3. This situation is displayed by assuming the higher temperature of the refrigerated liquid.

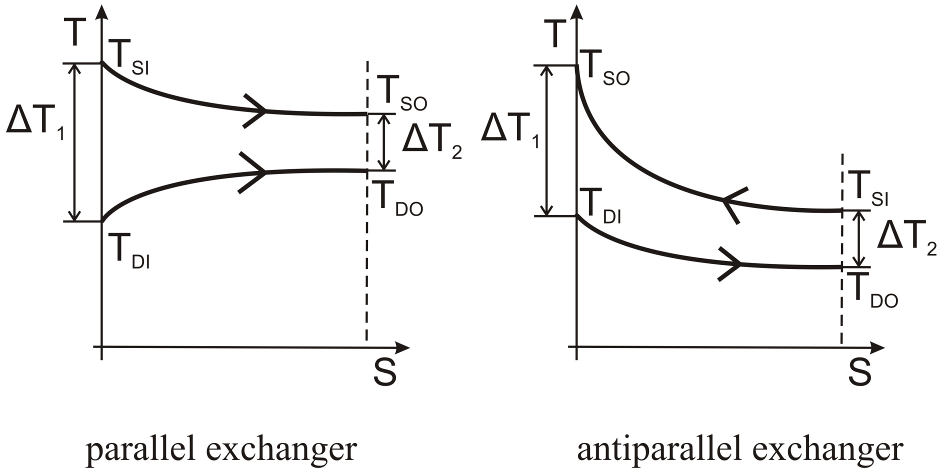

Fig. 7.3: The temperature profile of the parallel and antiparallel heat exchanger

The change of the heat flow rate dJ_{QS} lost by the refrigerated liquid is

$$dJ_{QS} = \dot{m}_S c_S dT_S,$$

where dT_S is the difference in temperatures, $\dot{m}_S$ is its mass rate. Similarly, the change of the heat flow rate for the coolant which receives the heat energy from refrigerated liquid

$$dJ_{QD} = \dot{m}_D c_D dT_D,$$

where dT_D is the difference in temperatures and its mass rate is $\dot{m}_D$. If $dT_D < 0$ then we talk about antiparallel compartment and for parallel arrangement we have $dT_D > 0$.

Since in the steady state it is satisfied that the heat energy lost by the warmer liquid has to be the same as the energy which the cooler liquid admits. Hence,

$$dJ_Q = -dJ_{QS} = \pm dJ_{QD}.$$

By substitution we obtain

$$dJ_Q = -\dot{m}_S c_S dT_S = \pm \dot{m}_D c_D dT_D. \tag{7.4}$$

It is satisfied for the change of the difference of the temperature

$$d(\Delta T) = d(T_S - T_D) = dT_S - dT_D, \tag{7.5}$$

where ΔT is the difference in temperatures of the refrigerated liquid T_S and the coolant T_D. This difference is a continuous function which depends on the space coordinate and thus it makes sense to talk about its derivative. Let us express (7.5) by using the substitution for dT_S and dT_D from (7.4)

$$d(\Delta T) = -\frac{dJ_Q}{\dot{m}_S c_S} \mp \frac{dJ_Q}{\dot{m}_D c_D} = dJ_Q\bigg(-\frac{1}{\dot{m}_S c_S} \mp \frac{1}{\dot{m}_D c_D}\bigg).$$

The heat flow rate is given as

$$J_Q = u\Delta T dS.$$

The change of difference in temperatures can be expressed as

$$d(\Delta T) = u\Delta T dS\bigg(-\frac{1}{\dot{m}_S c_S} \mp \frac{1}{\dot{m}_D c_D}\bigg).$$

This differential equation can be solved and the obtained solution is

$$\ln\frac{\Delta T_2}{\Delta T_1} = uS\bigg(-\frac{1}{\dot{m}_S c_S} \mp \frac{1}{\dot{m}_D c_D}\bigg) = uS\bigg(\frac{T_{SM} - T_{S0}}{J_Q} - \frac{T_{DM} - T_{D0}}{J_Q}\bigg).$$

The heat flow rate satisfies

$$J_Q = uS\Delta T_{TG},$$

where ΔT_{TG} is the middle logarithmic difference in temperatures which is given by the relation

$$\Delta T_{TG} = \frac{\Delta T_2 - \Delta T_1}{\ln\frac{\Delta T_2}{\Delta T_1}}.$$

Values ΔT_1 and ΔT_2 for the parallel and antiparallel heat exchanger follow from the figure 7.3 (see [19]).

8 The Efficiency of the Heat Exchanger

Once we have a heat exchanger we need to set some number for comparing which one is better in the sense of effectiveness. For this purpose we define efficiency E as a relationship between the actual heat transfer and the maximum possible heat transfer

$$E = \frac{\text{actual heat transfer}}{\text{maximum possible heat transfer}}. \tag{8.1}$$

In our notation we can rewrite the equation (8.1) in this form

$$E = \frac{j_Q}{j_{Q_opt}} = \frac{j_Q}{uS\left(\bar{T}_S - \bar{T}_D\right)}, \tag{8.2}$$

where S is the surface of the contact area with unit m^2 and u is the heat transfer coefficient with unit $\frac{W}{m^2K}$. (see [20]) It remains to say what $\bar{T}_S$ and $\bar{T}_D$ is. They are the average temperatures in the circuit of a refrigerated liquid for S and in the circuit of a coolant for D. We can define them as

$$\bar{T}_k = \frac{T_{kI} + T_{kO}}{2}, \qquad \text{where } k \in \{S, D\}.$$

At this point it remains to define the range of values for efficiency. It is easy to see from equation (8.1) that $E \geq 0$ and because of the physical point of view the maximum possible heat transfer is always greater than actual heat transfer there from it follows that $E \leq 1$. Consequently, $E \in [0; 1]$.

9 The Basic Types of Arrangements

In the previous chapters we summarized the theoretical assumptions of a heat transfer and now we will apply them to the particular case.

The heat exchanger is composed of three parts, two compartments and a contact area where the heat exchange occurs. One compartment is the compartment of refrigerated liquid and the second compartment is for the coolant. We will denote the compartment of refrigerated liquid as SC and the compartment of coolant as DC. The scheme of this exchanger is in the figure 9.1.

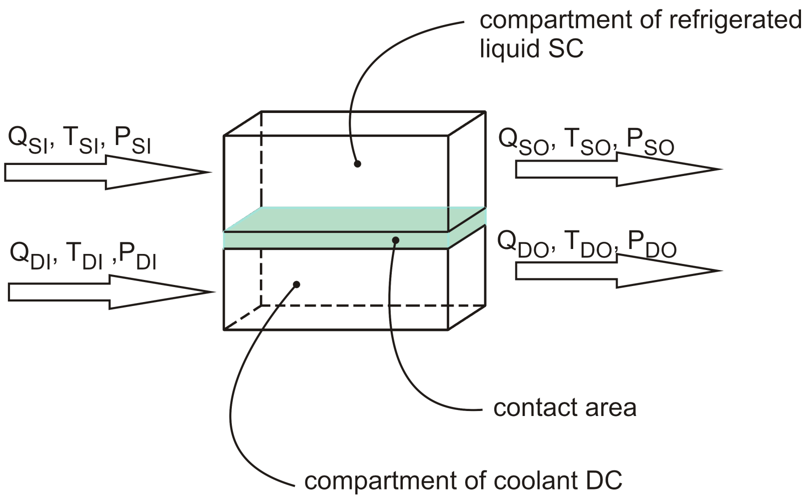

Fig. 9.1: The scheme of heat exchanger

We can observe that each compartment is characterized by six parameters, flow Q, temperature T and pressure P on the input I and output O. The sub-index S denotes the refrigerated liquid and D is for coolant. We say that the temperature of the coolant is known and controllable also its velocity profile is known. But its output temperature is measurable (see [5]). However, in the case of SC we do not know the temperature nor flow. Our goal is to find these parameters. That is why we need to solve the task as the inverse problem. We cannot do it directly because we do not have the precise description of SC. By that we mean initial and boundary conditions, velocity profile and all properties of refrigerated liquid connected with the heat transfer as the description of SC. Everything what we have is just a very precise description of DC and we want to derive some parameters of SC what is the typical inverse problem in partial differential equation. Its basic idea is to derive the rest of initial and boundary conditions which are missing from the measured data in a defined place.

We distinguish three basic arrangements with the respect to direction of the flow in SC and DC. The parallel flow arrangement occurs when the flow in SC and

DC have the angle of the direction equal to zero. If this angle is $\frac{\pi}{2}$ we talk about the cross flow arrangement. The antiparallel flow arrangement is defined for π. It is necessary to distinguish these situations because their mutual influence has an important impact to the final temperature. This is demonstrated in the figure 7.3 in the chapter heat transfer.

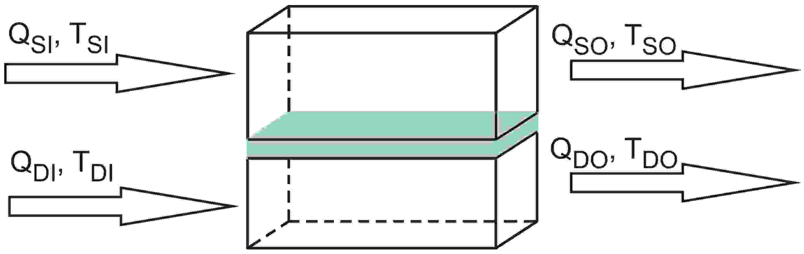

Fig. 9.2: The parallel flow arrangement

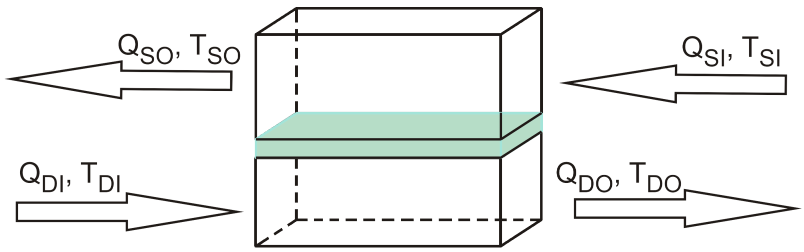

Fig. 9.3: The antiparallel flow arrangement

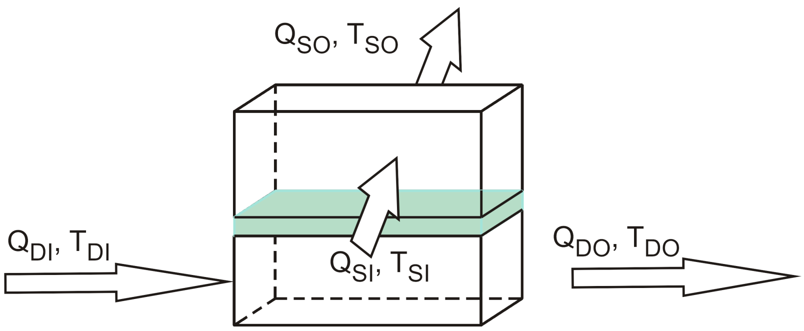

Fig. 9.4: The cross flow arrangement

We have to know the geometrical parameters of the exchanger to set the final equation of heat transfer. Their meaning is schematically shown in the following figure.

The geometrical parameters are not sufficient to solve the problem. We need to define also some conditions related to our problem to simplify the task.

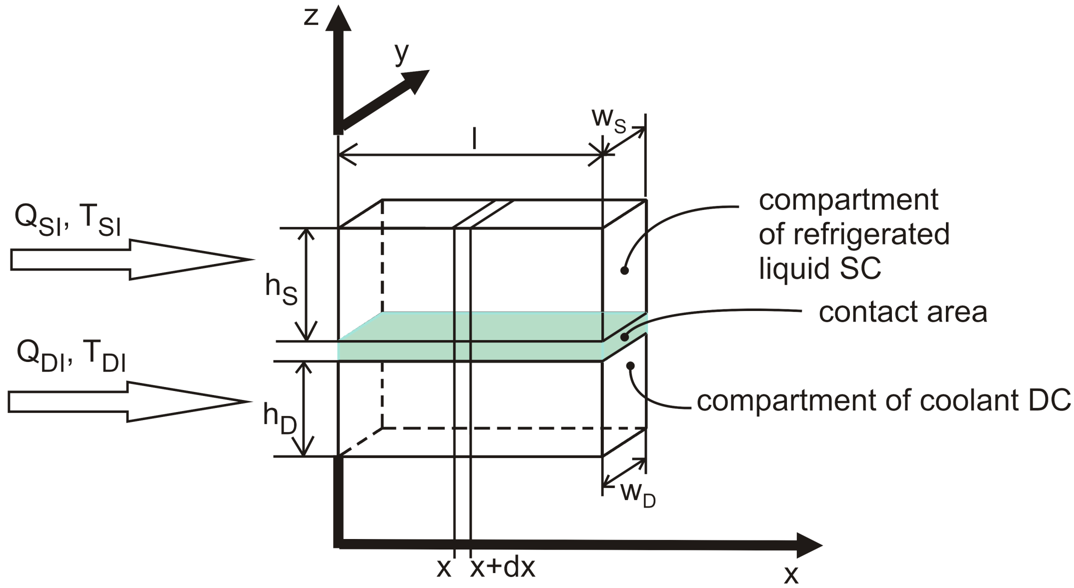

Fig. 9.5: The geometrical parameters of heat exchanger

1. At first we suppose the condition of incompressibility for both liquid.
2. The liquid cannot accumulate nor disappear inside the system. That means the heat exchanger is a closed system. Hence, $Q_{DI} = Q_{DO} = Q_D$, $Q_{SI} = Q_{SO} = Q_S$. (It was derived in the section 5.2.)
3. The heat exchange is done across the contact area.
4. Our system is closed with respect to energy. (It was derived in the section 5.4.)
5. The area of heat exchange is the same for the refrigerated liquid and the coolant. $w_S = w_D = w$
6. The liquid has a constant velocity profile in $w \times h_S$ respectively $w \times h_D$. There are no pressure gradients.
7. The temperature is constant in the whole volume $dx \times w \times h_S$ respectively $dx \times w \times h_D$. There is no polarization in the direction of axis z.
8. The parameters of the exchanger are homogeneous and time independent.

Hence, we write the first condition as

$$Q_{SI} + Q_{DI} = Q_{SO} + Q_{DO}.$$

The condition 4 provides no loss of energy so

$$Q_{SI}T_{SI}c_S + Q_{DI}T_{DI}c_D = Q_{SO}T_{SO}c_S + Q_{DO}T_{DO}c_D.$$

The condition 6 is very powerful and it is difficult to satisfy it. The velocity profile depends on many factors such as density, temperature, flow, ... The temperature of heat exchanger changes and so does the velocity profile. The last condition 8 expresses that thermal conductivity and geometrical dimensions do not depend on the position and time. The input and output temperature has a constant profile in the plane $w \times h_S$ and $w \times h_D$ for $x = 0$.

We formulate the balance equation for the refrigerated liquid and for the coolant and for $\varepsilon > 0$ under these conditions.

$$\begin{aligned}\frac{\partial T_S(x,t)}{\partial t} dxwh_S c_S \rho_S = &\underbrace{Q_{SI} T_S(t,x) c_S \rho_S}_{1} - \underbrace{Q_{SO} T_S(t, x+dx) c_S \rho_S}_{2} \\ &- \underbrace{udxw\Big(T_S(t, x+\varepsilon\ dx) - T_D(t, x+\varepsilon\ dx)\Big)}_{3},\end{aligned} \tag{9.1}$$

$$\begin{aligned}\frac{\partial T_D(x,t)}{\partial t} dxwh_D c_D \rho_D = &\underbrace{Q_{DI} T_D(t,x) c_D \rho_D}_{1} + \underbrace{Q_{DO} T_D(t, x+dx) c_D \rho_D}_{2} \\ &+ \underbrace{udxw\Big(T_S(t, x+\varepsilon\ dx) - T_D(t, x+\varepsilon\ dx)\Big)}_{3}.\end{aligned} \tag{9.2}$$

The term on the left-hand side of (9.1) and (9.2) describes the change of heat energy in the volume $dxwh_S = dV_S$. This change is made by terms on the right-hand side. We denoted three terms there. First term describes the amount of the heat accepted by the volume dV_S by convection. The second term gives us the amount of heat given by the volume dV_S by convection. The third one denotes the amount of heat transmitted into the coolant crossover the surface dxw where u is the overall heat transfer coefficient. We can apply the similar description on the equation (9.2).

We can simplify (9.1) and (9.2) under the assumption the condition 2. Thus $Q_{DI} = Q_{DO} = Q_D$, $Q_{SI} = Q_{SO} = Q_S$. Hence,

$$\begin{aligned}\frac{\partial T_S}{\partial t} &= -\frac{Q_S}{wh_S} \cdot \frac{\partial T_S}{\partial x} - \frac{u}{h_S \rho_S c_S}(T_S - T_D), \\ \frac{\partial T_D}{\partial t} &= -\frac{Q_D}{wh_D} \cdot \frac{\partial T_D}{\partial x} + \frac{u}{h_D \rho_D c_D}(T_S - T_D),\end{aligned}$$

where $\frac{Q_S}{wh_S}$, $\frac{Q_D}{wh_D}$ are velocities of liquids close to the contact area and $T_S = T_S(x,t)$, $T_D = T_D(x,t)$. If the condition 6 is satisfied then the velocity is independent on the axis z and y.

10 The Solution of the Antiparallel Flow Arrangement

The cross flow arrangement was already solved in the bachelor thesis (see [1]). There we supposed that the time derivative is equal to zero. Now we will deal with the parallel and the antiparallel flow arrangement where we cannot neglect the time derivative. So, we get a much more difficult problem.

Our aim is to solve the situation when we know the temperature of the coolant on the input $x = 0$ and on the output $x = l$. Because it allows us to define the temperature of the refrigerated liquid without a direct measuring in its circuit.

The equation which characterizes our system was introduced in the previous chapter. Now, we want to simplify it. For the sake of completeness let us rewrite it.

$$
\begin{aligned}
\frac{\partial T_S(t,x)}{\partial t} &= -\frac{Q_S}{wh_S} \cdot \frac{\partial T_S(t,x)}{\partial x} - \frac{u}{h_S \rho_S c_S}(T_S(t,x) - T_D(t,x)), \\
\frac{\partial T_D(t,x)}{\partial t} &= -\frac{Q_D}{wh_D} \cdot \frac{\partial T_D(t,x)}{\partial x} + \frac{u}{h_D \rho_D c_D}(T_S(t,x) - T_D(t,x)).
\end{aligned} \tag{10.1}
$$

The first equation describes the transport of heat from the cooled compartment to the cooling compartment. It expresses the equilibrium between decrease of heat in the infinitesimal volume $wh_S dx$ of sample due to transport through separation membrane to infinitesimal volume of cooling compartment $wh_D dx$. From that follows $T_S(t,x)$ has to be twice continuously differentiable function for $t \in \langle 0, \infty),\ x \in \langle 0, l\rangle$. Similarly, the second equation describes the transport of heat into cooling compartment from the cooled compartment. It expresses the equilibrium between increase of heat in the infinitesimal volume $wh_D dx$ of cooling liquid due to transport of heat through the separation membrane from infinitesimal volume $wh_S dx$ of cooled compartment.

The equation (10.1) with initial and boundary conditions describes our system. Without loss of generality we put the initial conditions equal to zero.

$$
\begin{aligned}
T_S(0,x) &= 0, \\
T_D(0,x) &= 0, \\
\frac{\partial}{\partial t} T_D(t,x)\Big|_{t=0} &= -\frac{Q_D}{wh_D}\frac{\partial}{\partial x} T_D(t,x)\Big|_{t=0}, \\
\frac{\partial}{\partial t} T_S(t,x)\Big|_{t=0} &= -\frac{Q_S}{wh_S}\frac{\partial}{\partial x} T_S(t,x)\Big|_{t=0}.
\end{aligned}
$$

The boundary conditions are

$$\begin{aligned} T_D(t,0) &= T_{DI}(t), \\ T_D(t,l) &= T_{DO}(t), \end{aligned}$$

where we are able to control the input temperature T_{DI} of the coolant and its output temperature T_{DO}.

We can introduce (10.2) because the $T_S(t,x)$ has the null initial condition. That is what enables to us to solve the problem because we will obtain the exactly defined boundary value problem for the unknown $T_D(t,x)$. First of all we will express $T_S(t,x)$ from the second equation of (10.1) and we get

$$T_S(t,x) = -\frac{\frac{\partial}{\partial t}T_D(t,x)wh_D\rho_Dc_D + Q_D\frac{\partial}{\partial x}T_D(t,x)\rho_Dc_D - uwT_D(t,x)}{uw}. \tag{10.2}$$

Then we substitute the $T_S(t,x)$ in the first equation of (10.1) with (10.2) and we will get the partial differential equation of the second order.

$$\begin{aligned} &\frac{h_D\rho_Dc_D}{u}\frac{\partial^2}{\partial t^2}T_D(t,x) + \left(\frac{Q_D\rho_Dc_D}{uw} + \frac{Q_Sh_D\rho_Dc_D}{wh_Su}\right)\frac{\partial^2}{\partial t\partial x}T_D(t,x) + \\ &+\left(1 + \frac{h_D\rho_Dc_D}{h_S\rho_Sc_S}\right)\frac{\partial}{\partial t}T_D(t,x) - \frac{(u\rho_Dc_DwQ_D + Q_S\rho_Sc_Suw)}{w^2h_Su\rho_Sc_S}\frac{\partial}{\partial x}T_D(t,x) + \\ &+\frac{Q_SQ_D\rho_Dc_D}{w^2h_Su}\frac{\partial^2}{\partial x^2}T_D(t,x) = 0 \end{aligned} \tag{10.3}$$

Now we can simplify (10.3) by substituting the coefficients as

$$\begin{aligned} &\alpha\frac{\partial}{\partial t}T_D(t,x) + \beta\frac{\partial}{\partial x}T_D(t,x) + \gamma\frac{\partial^2}{\partial t\partial x}T_D(t,x) + \\ &+\delta\frac{\partial^2}{\partial t^2}T_D(t,x) + \varepsilon\frac{\partial^2}{\partial x^2}T_D(t,x) = 0, \end{aligned} \tag{10.4}$$

where the coefficients $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ are defined as

$$\begin{aligned} &\alpha = 1 + \frac{\rho_Dc_Dh_D}{h_S\rho_Sc_S},\ \beta = -\frac{u\rho_Dc_DwQ_D + Q_S\rho_Sc_Suw}{w^2h_Su\rho_Sc_S}, \\ &\gamma = \frac{Q_D\rho_Dc_D}{uw} + \frac{Q_Sh_D\rho_Dc_D}{wh_Su},\ \delta = \frac{\rho_Dc_Dh_D}{u},\ \varepsilon = \frac{Q_SQ_D\rho_Dc_D}{w^2h_Su}. \end{aligned} \tag{10.5}$$

It is very important to note that there is a relation between coefficients $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ given as

$$\varepsilon\alpha^2\delta - 4\varepsilon\alpha\delta + \gamma^2\alpha - \delta\beta\gamma\alpha + 4\varepsilon\delta - \gamma^2 + \delta^2\beta^2 = 0. \tag{10.6}$$

This defines the identifiability of the original parameters. This condition is so called compatibility condition. We can verify the relation. We know from (10.5) ε and also we have from (10.6) that

$$\varepsilon = \frac{(-\gamma + \delta\beta)(\gamma\alpha - \gamma - \delta\beta)}{\delta(\alpha-2)^2}.$$

Then by the direct substitution we receive the equality

$$-\frac{Q_S Q_D \rho_D c_D}{w^2 h_S u} = -\frac{Q_S Q_D \rho_D c_D}{w^2 h_S u}.$$

Our problem is rather complex so we will divide it in two parts the stationary and the non-stationary one.

10.1 The Stationary State

We suppose that time variation are very slow in the stationary state so we can consider the time as parameter. So, we can neglect the time derivative and put it equal to zero. We obtain

$$\begin{aligned} -\frac{Q_S}{wh_S}\cdot\frac{\partial T_S(t,x)}{\partial x} - \frac{u}{h_S\rho_S c_S}(T_S(t,x) - T_D(t,x)) &= 0, \\ -\frac{Q_D}{wh_D}\cdot\frac{\partial T_D(t,x)}{\partial x} + \frac{u}{h_D\rho_D c_D}(T_S(t,x) - T_D(t,x)) &= 0. \end{aligned}$$

Since, we do not have the time derivative we can substitute $T_D(t,x) = T_D(x)$ and similarly $T_S(t,x) = T_S(x)$. Then we obtain the boundary value problem

$$\begin{aligned} -\frac{Q_S}{wh_S}\cdot\frac{dT_S(x)}{dx} - \frac{u}{h_S\rho_S c_S}(T_S(x) - T_D(x)) &= 0, \\ -\frac{Q_D}{wh_D}\cdot\frac{dT_D(x)}{dx} + \frac{u}{h_D\rho_D c_D}(T_S(x) - T_D(x)) &= 0, \\ T_D(0) &= T_{DI}(t), \\ T_D(l) &= T_{DO}(t). \end{aligned} \tag{10.7}$$

Hence, we get the solution

$$\begin{aligned}
T_D(x) &= \frac{\mathrm{e}^{-\vartheta l} T_{DI}(t) - T_{DO}(t)}{-1+\mathrm{e}^{-\vartheta l}} - \\
&\quad - \frac{\left(T_{DI}(t) - T_{DO}(t)\right) \mathrm{e}^{-\vartheta x}}{-1+\mathrm{e}^{-\vartheta l}}, \\
T_S(x) &= -\frac{1}{Q_S \rho_S c_S} \left(\frac{\left(\mathrm{e}^{-\vartheta l} T_{DI}(t) - T_{DO}(t)\right) Q_S \rho_S c_S}{-1+\mathrm{e}^{-\vartheta l}} - \right. \\
&\quad \left. - \frac{Q_D \rho_D c_D \left(T_{DI}(t) - T_{DO}(t)\right) \mathrm{e}^{-\vartheta x}}{-1+\mathrm{e}^{-\vartheta l}} \right),
\end{aligned}$$

where

$$\vartheta = \frac{uw}{Q_S \rho_S c_S} + \frac{uw}{Q_D \rho_D c_D}.$$

Our aim was find the $T_S(0, x)$ and $T_S(l, x)$ from the $T_{DO}(t)$. After simplification and backward transformation we receive the following

$$\begin{aligned}
T_S(0,t) &= \frac{\left(Q_S \rho_S c_S \mathrm{e}^{-\vartheta l} + Q_D \rho_D c_D\right) T_{DI}(t)}{\left(-1+\mathrm{e}^{-\vartheta l}\right) Q_S \rho_S c_S} + \\
&\quad + \frac{\left(-Q_S \rho_S c_S - Q_D \rho_D c_D\right) T_{DO}(t)}{\left(-1+\mathrm{e}^{-\vartheta l}\right) Q_S \rho_S c_S}
\end{aligned} \tag{10.8}$$

and

$$\begin{aligned}
T_S(l,t) &= \frac{\left(Q_S \rho_S c_S \mathrm{e}^{-\vartheta l} + Q_D \rho_D c_D \mathrm{e}^{-\vartheta l}\right) T_{DI}(t)}{\left(-1+\mathrm{e}^{-\vartheta l}\right) Q_S \rho_S c_S} + \\
&\quad + \frac{\left(-Q_S \rho_S c_S - Q_D \rho_D c_D \mathrm{e}^{-\vartheta l}\right) T_{DO}(t)}{\left(-1+\mathrm{e}^{-\vartheta l}\right) Q_S \rho_S c_S}.
\end{aligned} \tag{10.9}$$

Also, if we suppose that time variations are very slow then we are able to determine the temperature profile in any point and at any time such as

$$\begin{aligned}
T_S(x,t) &= -\frac{\left(\frac{\mathrm{e}^{-\vartheta l} Q_S \rho_S c_S}{-1+\mathrm{e}^{-\vartheta l}} - \frac{Q_D \rho_D c_D \mathrm{e}^{-\vartheta x}}{-1+\mathrm{e}^{-\vartheta l}}\right) T_{DI}(t)}{Q_S \rho_S c_S} - \\
&\quad - \frac{\left(\frac{Q_S \rho_S c_S}{-1+\mathrm{e}^{-\vartheta l}} + \frac{Q_D \rho_D c_D \mathrm{e}^{-\vartheta x}}{-1+\mathrm{e}^{-\vartheta l}}\right) T_{DO}(t)}{Q_S \rho_S c_S}.
\end{aligned} \tag{10.10}$$

10.2 The Non-Stationary State

Our next steps are again based on the equation (10.1). For the better readability, let us rewrite

$$
\begin{aligned}
\frac{\partial T_S(t,x)}{\partial t} &= -\frac{Q_S}{wh_S}\cdot\frac{\partial T_S(t,x)}{\partial x} - \frac{u}{h_S\rho_S c_S}(T_S(t,x)-T_D(t,x)),\\
\frac{\partial T_D(t,x)}{\partial t} &= -\frac{Q_D}{wh_D}\cdot\frac{\partial T_D(t,x)}{\partial x} + \frac{u}{h_D\rho_D c_D}(T_S(t,x)-T_D(t,x)).
\end{aligned}
$$

Let us introduce the following substitution

$$
a_{11}=\frac{Q_S}{wh_S},\quad a_{12}=\frac{u}{h_S\rho_S c_S},\quad a_{21}=\frac{Q_D}{wh_D},\quad a_{22}=\frac{u}{h_D\rho_D c_D}. \tag{10.11}
$$

If the transformation (10.11) is satisfied, we define the inverse one for known parameters Q_D and h_D as

$$
u=a_{22}h_D\rho_D c_D,\quad w=\frac{Q_D}{a_{21}h_D},\quad Q_S=\frac{a_{11}Q_D a_{22}\rho_D c_D}{a_{21}a_{12}\rho_S c_S},\quad h_S=\frac{a_{22}h_D\rho_D c_D}{a_{12}\rho_S c_S}.
$$

Analogously, we obtain for known $Q_D,\ u$

$$
w=\frac{Q_D a_{22}\rho_D c_D}{a_{21}u},\quad Q_S=\frac{a_{11}Q_D a_{22}\rho_D c_D}{a_{21}a_{12}\rho_S c_S},\quad h_D=\frac{u}{a_{22}\rho_D c_D},\quad h_S=\frac{u}{a_{12}\rho_S c_S}.
$$

After substitution (10.11) we receive

$$
\frac{\partial}{\partial t}T_S(t,x) = -a_{11}\tfrac{\partial}{\partial x}T_S(t,x) - a_{12}(T_S(t,x)-T_D(t,x)), \tag{10.12}
$$

$$
\frac{\partial}{\partial t}T_D(t,x) = -a_{21}\tfrac{\partial}{\partial x}T_D(t,x) + a_{22}(T_S(t,x)-T_D(t,x)), \tag{10.13}
$$

where $t\in\langle 0,\infty),\ x\in\langle 0,l\rangle$. We eliminate the unknown $T_S(t,x)$ from the (10.12) and we get

$$
T_S(t,x)=\frac{\frac{\partial}{\partial t}T_D(t,x)+a_{21}\frac{\partial}{\partial x}T_D(t,x)+a_{22}T_D(t,x)}{a_{22}}.
$$

We obtain the final equation after substituting $T_S(t,x)$ in the (10.13) as

$$
\begin{aligned}
&\frac{\frac{\partial^2}{\partial t^2}T_D(t,x)}{a_{22}}+\frac{(a_{11}+a_{21})\frac{\partial^2}{\partial x\partial t}T_D(t,x)}{a_{22}}+\frac{a_{11}a_{21}\frac{\partial^2}{\partial x^2}T_D(t,x)}{a_{22}}+\\
&+\frac{(a_{22}+a_{12})\frac{\partial}{\partial t}T_D(t,x)}{a_{22}}+\frac{(a_{12}a_{21}+a_{11}a_{22})\frac{\partial}{\partial x}T_D(t,x)}{a_{22}}=0.
\end{aligned} \tag{10.14}
$$

Let us suppose that we know Q_D and u. Then we can transform the initial and boundary conditions

$$
\begin{aligned}
T_D(0,x) &= T_{DI}(x),\\
\frac{\partial}{\partial t}T_D(t,x)\Big|_{t=0} &= -a_{21}\frac{\partial}{\partial x}T_D(t,x)\Big|_{t=0} + a_{22}\left(T_{SI}(x) - T_{DI}(x)\right),\\
T_D(t,0) &= T_{DI}(t),\\
T_D(t,l) &= T_{DO}(t).
\end{aligned}
$$

Now we want to simplify the final equation (10.14). For that we will use the following transformation

$$
t = b_{11}\tau + b_{12}\xi, \quad x = b_{21}\tau + b_{22}\xi, \quad T_D(t,x) = AU(\tau,\xi)\mathrm{e}^{-p\tau}\mathrm{e}^{-q\xi}, \tag{10.15}
$$

where $p,\ q,\ b_{11},\ b_{12},\ b_{21},\ b_{22} \in \mathbb{R}$. The inverse transform is

$$
\begin{aligned}
\tau &= -\frac{xb_{12} - b_{22}t}{b_{11}b_{22} - b_{21}b_{12}}, \quad \xi = \frac{-tb_{21} + b_{11}x}{b_{11}b_{22} - b_{21}b_{12}},\\
U(\tau,\xi) &= \frac{T_D(t,x)\mathrm{e}^{-\frac{p(xb_{12}-b_{22}t)}{b_{11}b_{22}-b_{21}b_{12}} + \frac{q(-tb_{21}+b_{11}x)}{b_{11}b_{22}-b_{21}b_{12}}}}{A}.
\end{aligned}
$$

We obtain the complicated formula when we apply the transformation (10.15). Derivation was done in Maple. The file is attached.

The equation (10.14) can be transformed by (10.15) to the form

$$
-\frac{\partial^2}{\partial \tau^2}U(\tau,\xi) + \frac{\partial^2}{\partial \xi^2}U(\tau,\xi) - U(\tau,\xi) = 0, \tag{10.16}
$$

where coefficients $p,\ q,\ b_{12},\ b_{21},\ b_{22}$ of transformation are defined as

$$
\begin{aligned}
p &= \frac{b_{11}a_{22}a_{11} - b_{11}a_{12}a_{21} - a_{22}b_{21} + a_{12}b_{21}}{a_{11} - a_{21}},\\
q &= \frac{-b_{22}a_{22} + b_{22}a_{12} + a_{11}a_{22}b_{12} - a_{21}b_{12}a_{12}}{a_{11} - a_{21}},\\
b_{12} &= -\frac{\sqrt{a_{12}a_{22}\left(a_{12}a_{22}b_{11}^2 + 1\right)}}{a_{12}a_{22}},\\
b_{21} &= -\frac{1}{2}\frac{1}{a_{12}a_{22}}\left(a_{12}a_{22}\left(\begin{array}{l}
2a_{21}^2b_{11}^2a_{12}a_{22} + \\
+2a_{21}b_{11}\sqrt{a_{12}a_{22}(a_{12}a_{22}b_{11}^2+1)(a_{11}-a_{21})^2} + \\
+2a_{11}b_{11}\sqrt{a_{12}a_{22}(a_{12}a_{22}b_{11}^2+1)(a_{11}-a_{21})^2} + \\
+2a_{11}^2b_{11}^2a_{12}a_{22} + a_{11}^2 - 2a_{11}a_{21} + a_{21}^2
\end{array}\right)\right)^{\frac{1}{2}},\\
b_{22} &= \frac{b_{12}\left(2a_{11}a_{21}b_{11} - a_{21}b_{21} - a_{11}b_{21}\right)}{-2b_{21} + a_{21}b_{11} + a_{11}b_{11}}.
\end{aligned} \tag{10.17}
$$

The equation (10.16) is the Gordon-Klein equation which was already solved in the second chapter. A different way how to express its solution is in the form as sequence. It is important to remind that the whole procedure is based on the assumption of zero initial conditions.

$$U(\tau,\xi) = \sum_{n=0}^{\infty} C_n \mathrm{e}^{D_n(\xi-\tau)} \mathrm{e}^{\frac{1}{2}\frac{\xi+\tau}{D_n}}, \tag{10.18}$$

where $C_n,\ D_n \in \mathbb{R}$ for $n = 0, 1, \ldots, \infty$.

So, we transform the inverse problem to the problem of setting six parameters of transform which is much easier.

10.2.1 The Singular Case

As it was already commented, we receive the singular case for $a_{11} = a_{21}$. Thus,

$$\frac{Q_S}{wh_S} = \frac{Q_D}{wh_D}.$$

Hence,

$$Q_S = \frac{Q_D h_S}{h_D}.$$

So, we can simplify (10.1) as

$$\begin{aligned}
\frac{\partial T_S(t,x)}{\partial t} &= -\frac{Q_D}{wh_D} \cdot \frac{\partial T_S(t,x)}{\partial x} - \frac{u}{h_S \rho_S c_S}(T_S(t,x) - T_D(t,x)),\\
\frac{\partial T_D(t,x)}{\partial t} &= -\frac{Q_D}{wh_D} \cdot \frac{\partial T_D(t,x)}{\partial x} + \frac{u}{h_D \rho_D c_D}(T_S(t,x) - T_D(t,x)).
\end{aligned}$$

Then we subtract the previous equations and we obtain

$$\begin{aligned}
\frac{\partial T_S(t,x)}{\partial t} - \frac{\partial T_D(t,x)}{\partial t} = -\frac{Q_D}{wh_D} \cdot \frac{\partial T_S(t,x)}{\partial x} + \frac{Q_D}{wh_D} \cdot \frac{\partial T_D(t,x)}{\partial x} - \\
- \frac{u(T_S(t,x) - T_D(t,x))}{h_S \rho_S c_S} - \frac{u(T_S(t,x) - T_D(t,x))}{h_D \rho_D c_D}.
\end{aligned}$$

Now we define the transformation $T_D(t,x) = V(t,x) + T_S(t,x)$ and we obtain

$$-\frac{\partial V(t,x)}{\partial t} = \frac{uV(t,x)}{h_S \rho_S c_S} + \frac{Q_D \frac{\partial V(t,x)}{\partial x}}{h_D w} + \frac{uV(t,x)}{h_D \rho_D c_D}.$$

The solution $V(t,x)$ is

$$V(t,x) = \left(T_{DI}\left(\frac{xwh_D - Q_D t}{wh_D}\right) - T_{SI}\left(\frac{xwh_D - Q_D t}{wh_D}\right)\right) \mathrm{e}^{-\kappa t},$$

where

$$\kappa = \frac{u\left(h_D \rho_D c_D + h_S \rho_S c_S\right)}{h_S \rho_S c_S h_D \rho_D c_D}.$$

Hence, we derive the solution $T_S(t,x)$ for the singular case as

$$\begin{aligned} T_S(t,x) &= -\left(T_{DI}\left(\frac{xwh_D - Q_D t}{wh_D}\right) - T_{SI}\left(\frac{xwh_D - Q_D t}{wh_D}\right)\right)\mathrm{e}^{-\kappa t} + \\ &\quad + T_D(t,x). \end{aligned} \tag{10.19}$$

10.3 The Homogeneous Boundary Conditions

Now we will deal with the homogeneous boundary conditions to obtain the Gordon-Klein equation in the same form as in Subchapter 2.3. We assume the same example as above. To guarantee the homogeneous boundary conditions we introduce $T_D(t,x)$ in the following form

$$T_D(t,x) = \frac{T_{DO}(t)x}{l} + \frac{T_{DI}(t)(l-x)}{l} + W(t,x). \tag{10.20}$$

Thus, the boundary conditions are

$$\begin{aligned} T_D(t,0) &= T_{DI}(t) + W(t,0), \\ T_D(t,l) &= T_{DO}(t) + W(t,l). \end{aligned}$$

Hence,

$$W(t,0) = 0, \tag{10.21}$$

$$W(t,l) = 0. \tag{10.22}$$

Then we evaluate (10.14) with (10.20) and we obtain

$$\begin{aligned} &\frac{\frac{\partial^2}{\partial t^2}W(t,x)}{a_{22}} + \frac{(a_{11}+a_{21})\frac{\partial^2}{\partial x \partial t}W(t,x)}{a_{22}} + \frac{a_{11}a_{21}\frac{\partial^2}{\partial x^2}W(t,x)}{a_{22}} + \\ &+\frac{(a_{22}+a_{12})\frac{\partial}{\partial t}W(t,x)}{a_{22}} + \frac{(a_{12}a_{21}+a_{11}a_{22})\frac{\partial}{\partial x}W(t,x)}{a_{22}} = P(x,t). \end{aligned} \tag{10.23}$$

where

$$
\begin{aligned}
P(x,t) &= -\frac{x\frac{d^2}{dt^2}T_{DO}(t)}{a_{22}l}+\left(\frac{x}{a_{22}l}-\frac{1}{a_{22}}\right)\frac{d^2}{dt^2}T_{DI}(t)- \\
&-\left(\frac{(a_{12}+a_{22})x}{a_{22}l}+\frac{a_{11}+a_{21}}{a_{22}l}\right)\frac{d}{dt}T_{DO}(t)- \\
&-\left(\frac{(-a_{12}-a_{22})x}{a_{22}l}+\frac{a_{12}+a_{22}}{a_{22}}-\frac{a_{21}+a_{11}}{a_{22}l}\right)\frac{d}{dt}T_{DI}(t)- \\
&-\frac{(a_{12}a_{21}+a_{11}a_{22})T_{DO}(t)}{a_{22}l}-\frac{(-a_{12}a_{21}-a_{11}a_{22})T_{DI}(t)}{a_{22}l}.
\end{aligned}
\tag{10.24}
$$

We receive the following in the boundary points

$$
\begin{aligned}
P(0,t) &= -\frac{1}{a_{22}}\frac{d^2}{dt^2}T_{DI}(t)-\frac{a_{11}+a_{21}}{a_{22}l}\frac{d}{dt}T_{DO}(t)- \\
&-\left(\frac{a_{12}+a_{22}}{a_{22}}-\frac{a_{21}+a_{11}}{a_{22}l}\right)\frac{d}{dt}T_{DI}(t)- \\
&-\frac{(a_{12}a_{21}+a_{11}a_{22})T_{DO}(t)}{a_{22}l}-\frac{(-a_{12}a_{21}-a_{11}a_{22})T_{DI}(t)}{a_{22}l}, \\
P(l,t) &= -\frac{\frac{d^2}{dt^2}T_{DO}(t)}{a_{22}}-\left(\frac{(a_{12}+a_{22})}{a_{22}}+\frac{a_{11}+a_{21}}{a_{22}l}\right)\frac{d}{dt}T_{DO}(t)- \\
&-\left(\frac{(-a_{12}-a_{22})}{a_{22}}+\frac{a_{12}+a_{22}}{a_{22}}-\frac{a_{21}+a_{11}}{a_{22}l}\right)\frac{d}{dt}T_{DI}(t)- \\
&-\frac{(a_{12}a_{21}+a_{11}a_{22})T_{DO}(t)}{a_{22}l}-\frac{(-a_{12}a_{21}-a_{11}a_{22})T_{DI}(t)}{a_{22}l}.
\end{aligned}
$$

When we apply the same procedure as above to this problem we receive

$$
-\frac{\partial^2}{\partial\tau^2}U(\tau,\xi)+\frac{\partial^2}{\partial\xi^2}U(\tau,\xi)-U(\tau,\xi)=\frac{P\left(\phi(\xi),\varphi(\xi,\tau)\right)}{\mathrm{e}^{\psi(\xi,\tau)}Aa_{12}}. \tag{10.25}
$$

where

$$
\begin{aligned}
\phi(\xi) &= \frac{\sqrt{a_{11}a_{21}}\xi}{\sqrt{a_{12}a_{22}}}, \\
\varphi(\xi,\tau) &= \frac{1}{2}\frac{\tau a_{11}-\tau a_{21}+\xi a_{11}+\xi a_{21}}{\sqrt{a_{12}a_{11}a_{21}a_{22}}}, \\
\psi(\xi,\tau) &= \frac{1}{2}\frac{\tau a_{11}a_{22}-\tau a_{12}a_{21}+\xi a_{11}a_{22}+\xi a_{12}a_{21}}{\sqrt{a_{12}a_{11}a_{21}a_{22}}}.
\end{aligned}
$$

The used transformation was the same like previous, i.e. (10.15) and the obtained coefficients are

$$
\begin{aligned}
p &= \frac{1}{2}\frac{a_{22}a_{11}-a_{12}a_{21}}{\sqrt{a_{12}a_{11}a_{21}a_{22}}},\\
q &= \frac{1}{2}\frac{a_{22}a_{11}+a_{12}a_{21}}{\sqrt{a_{12}a_{11}a_{21}a_{22}}},\\
b_{11} &= \frac{1}{4}\sqrt{4}(a_{11}-a_{21})\sqrt{\frac{1}{a_{12}a_{11}a_{21}a_{22}}},\\
b_{12} &= \frac{1}{2}\frac{a_{11}+a_{21}}{\sqrt{a_{12}a_{22}a_{21}a_{11}}},\\
b_{21} &= 0,\\
b_{22} &= \sqrt{\frac{a_{11}a_{21}}{a_{12}a_{22}}}.
\end{aligned}
$$

The Green function for the Gordon-Klein equation (10.25) with homogeneous boundary conditions (10.21), (10.22) is defined as

$$
G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}\left(\sin\left(\frac{\pi n}{l}x\right)\sin\left(\frac{\pi n}{l}\xi\right)\frac{\sin\left(t\sqrt{\frac{\pi^2n^2}{l^2}+1}\right)}{\sqrt{\frac{\pi^2n^2}{l^2}+1}}\right) \tag{10.26}
$$

(see [21]). The appropriate solution is given as

$$
W(t,x)=\int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t)d\xi d\tau, \tag{10.27}
$$

where

$$
\Phi(\xi,\tau)=-\frac{P\left(\phi(\xi),\varphi(\xi,\tau)\right)}{\mathrm{e}^{\psi(\xi,\tau)}Aa_{12}}. \tag{10.28}
$$

From now we are able to derive the temperature profile in any point at any time in the heat exchanger.

10.4 The Parallel Flow Arrangement

The scheme of the parallel flow arrangement is shown in the figure 9.2. We can observe that the refrigerated liquid enters in the system in the same position as the coolant. That is why we formulate boundary conditions as

$$
\begin{aligned}
T_S\left(t,0\right) &= T_{SI}\left(t\right), \\
T_D\left(t,0\right) &= T_{DI}\left(t\right), \\
\frac{\partial}{\partial t}T_S\left(t,x\right)\Big|_{x=0} &= -\frac{Q_S}{wh_S}\frac{\partial}{\partial x}T_S\left(t,x\right)\Big|_{x=0} + \frac{u\left(T_{SI}\left(t\right)-T_{DI}\left(t\right)\right)}{h_S\rho_S c_S}, \\
\frac{\partial}{\partial t}T_D\left(t,x\right)\Big|_{x=0} &= -\frac{Q_D}{wh_D}\frac{\partial}{\partial x}T_D\left(t,x\right)\Big|_{x=0} - \frac{u\left(T_{SI}\left(t\right)-T_{DI}\left(t\right)\right)}{h_D\rho_D c_D}.
\end{aligned}
$$

The parallel flow arrangement is a very similar case to the antiparallel. The only difference is that the refrigerated liquid has the opposite direction. So, we get the solution for the parallel flow arrangement if we use $-Q_S$ instead of Q_S into the solution of the antiparallel one.

11 The Results and Discussion

In this thesis, we have studied the problems connected with the heat transfer, the identifiability, controllability and observability of a parameter. In particular, we have defined the basic types of the arrangement of the heat exchanger.

We introduced the mathematical model describing our system. This model is characterized by the input parameter T_{DI}, which is controllable, by the output parameter T_{DO}, which is observable and identifiable, and by the parameters T_{SI}, T_{SO}, which we wished to identify.

Since the problem was rather complex, we solved it in several stages, namely the stationary state, the non-stationary one and the case of homogeneous conditions.

The stationary state is described via (10.7). It is important to mention that we supposed that the time variations are significantly slower than processes connected with the heat transfer inside the heat exchanger. Thus, we could neglect them. So, we received the boundary value problem for the ordinary differential equation. Our aim was to define the input and output temperature of the refrigerated liquid which are given by the equations (10.8) and (10.9). Under the above mentioned assumption, we were able to determine the temperature T_S in cooled circuit in any point at any time via (10.10).

Subsequently, we defined the stationary state solution and continued with the non-stationary processes. In this process, we could not neglect the time derivatives because the coolant and the refrigerated liquid affect each other in time (see Fig. 7.3). This phenomenon follows from the fact that the liquids flows along the contact simultaneously. This non-stationary case is described by (10.14) which is the partial differential equation of the second order describing the cooling compartment. It was possible to transform it to the simple Gordon-Klein equation (10.16) by using the transformation (10.15). Thus we get the general coefficients $b_{11},\ b_{12},\ b_{21},\ b_{22}$ and $p,\ q$. The relations for them are given by (10.17). These coefficients are arbitrary and we used this property to eliminate some of the terms in the given equation. Sometimes it is possible to solve this problem by separation of variables, especially for the homogeneous boundary conditions. In this situation, it is necessary to use the solution of the equation (10.16) in the form of infinite series (10.18). This solution allows us to determine the temperature T_S in cooled circuit in any point at any time. We obtained the singular case for $a_{11} = a_{21}$, which we solved individually. In this case it is possible to express the solution in the form (10.19).

In Subchapter 10.3, we dealt with the transformation of our problem to the problem with homogeneous boundary conditions. We introduced the solution of this case by the Green function (10.26) and we expressed it by relations (10.27), (10.28), (10.24) again for any point at any time.

Also, we received the solution for the parallel arrangement if we substituted Q_S by $-Q_S$. Thus, we achieved the change of the direction which is the only one difference between the parallel and antiparallel arrangement.

We were successful and gained the formal solution of the given problem. That allows us to solve the inverse problem in such a way that measured data on the output T_{DO} (which are observable) of the heat exchanger are possible to be defined for the appropriate setting of the parameters on the input T_{DI} (these are controllable for example $T_{DI} = 0$) by using of appropriate optimization method. For example, it is possible to approximate the measured data by derived relationships in which the unknown parameters are estimated by the least squares method or the maximum likelihood method (for more details see [22]). We have to mention that parameters are included in non-linear functions, e.g. in the exponential function. Therefore, the condition of the normal distribution of the errors of measuring in the experimental data is not satisfied. Hence, methods based on the maximum likelihood method will probably give better results for estimation of parameters.

The importance of getting solution by using the compatibility conditions (10.6) is following. Let us suppose that we do not have the compatibility conditions. Then we have to derive the appropriate coefficient from the equation (10.25). That leads to the bigger noise and thus also higher error. The equation (10.6) says that parameters (10.5) has the internal relation expressed by (10.6). That means that if we would not consider this relation and if we would solve the given problem only numerically we would get the solution which would be random and define by rounding errors of numerical computations.

So, we converted the character of the task from finding the solution of the given problem to finding the six parameters $b_{11},\ b_{12},\ b_{21},\ b_{22}$ and $p,\ q$ of the transformation, which is much easier to solve.

12 Conclusion

The aim of this thesis was to summarize the knowledge about the heat transfer and determine the solution of heat exchange for some particular cases. We dealt with the antiparallel heat exchanger, where the input temperature of the coolant T_{DI} is controllable. Its output T_{DO} is not controllable because the coolant is affected by the presence of refrigerated liquid. However, we can still measure this output temperature T_{DO}, hence it is observable. We wished to define the input and output temperature of the cooled circuit T_{SI} and T_{SO}. These parameters are not controllable nor observable, but identifiable.

Further, we solved our problem separately for the stationary state and the non-stationary state. We obtained the temperature profile at any time and any point of the heat exchanger for each particular case. Thus, we simplified the character of the task.

Hence, we are able to identify the temperature T_S in the cooled circuit at any point and any time for the antiparallel arrangement of the heat exchanger without a real measuring in the circuit. This can be very useful in the situation, when we do not have the access to the cooled circuit at arbitrary time. Also, it can happen that the refrigerated liquid has some "bad" properties such as radiation, corrosivity, acidity, etc.

The contribution of this thesis also follows from the fact that we are able to set the temperature profile of T_S at any place and any time. It can be used to define the position of a failure of machine, which is convenient with respect to its maintenance and repair.

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List of symbols, physical constants and abbreviations

Q_{SI} input flow of refrigerated liquid

T_{SI} input temperature of refrigerated liquid

Q_{SO} output flow of refrigerated liquid

T_{SO} output temperature of refrigerated liquid

$\bar{T}_S$ average temperature of refrigerated liquid

Q_{DI} input flow of coolant

T_{DI} input temperature of coolant

Q_{DO} output flow of coolant

T_{DO} output temperature of coolant

$\bar{T}_D$ average temperature of coolant

u overall heat transfer coefficient

a thermal diffusivity

S surface of a contact area

J_Q heat flux

j_Q specific heat flux

ρ density

Ω bounded domain which is occupied by the liquid

$\mathbf{x}$ any point of Ω

$V(t)$ control volume at time t

t time instant from interval $[0, T]$ where $T > 0$

E efficiency

$\mathbf{v}$ velocity of flow

$\mathbf{n}$ normal vector

μ dynamic viscosity

ν kinematic viscosity

Re Reynolds number

v_M mean velocity

ϕ_H hydraulic diameter

A cross sectional area

P wetted perimeter of the cross-section

δ thickness of boundary layer

T_{SM} temperature of refrigerated liquid which is not influenced by boundary layer

T_{S1} temperature of refrigerated liquid in the boundary liquid

c_S heat capacity of refrigerated liquid

α_S individual convection heat transfer coefficient of refrigerated liquid

w_S width of refrigerated liquid compartment

h_S height of refrigerated liquid compartment

ρ_S density of refrigerated liquid

δ_S thickness of boundary layer of refrigerated liquid

T_{DM} temperature of coolant which is not influenced by boundary layer

T_{D1} temperature of coolant in the boundary liquid

c_D heat capacity of coolant

α_D individual convection heat transfer coefficient of coolant

w_D width of coolant compartment

h_D height of coolant compartment

ρ_D density of coolant

δ_D thickness of boundary layer of coolant

k_D parameter $k_D = \frac{u}{c_D \rho_D}$

T_M mean temperature of liquid

T_W temperature of wall

α individual convection heat transfer coefficient

$\dot{m}_S$ mass rate of refrigerated liquid

$\dot{m}_D$ mass rate of coolant

R heat resistance

Φ shape of body

λ thermal conductivity

λ_W thermal conductivity of a contact area

w width

h height

l length

c heat capacity per particle

d_0 thickness of contact area

d thickness of contact area and boundary layers ($d = \delta_S + d_0 + \delta_D$)

dJ_{QS} change of heat flow rate of refrigerated liquid

dJ_{QD} change of heat flow rate of coolant

v velocity of flow

DC coolant compartment

SC compartment of refrigerated liquid